MATHEMATICS IN THE CHRISTIAN SCHOOL

- a preliminary report resulting from a seminar held in Toronto in July, 1971, under the auspices of the O.A.C.S. and the A.A.C.S.

LIST OF PARTICIPANTS IN THE MATH SEMINAR, SUMMER 1971

Mr. Andries Batelaan, Group 1, Box 44, RR#6, Bowmanville, Ont.

Mr. Gerry De Koning, 20 Crispin Crescent, Willowdale, Ont.

Mrs. Karen Greenham, Beacon Christian High School, 99 Scott Street, St. Catharines, Ont.

Mr. Rick Jongejan, 302 Wolfe Street, Apt. 2, London 14, Ont.

Mrs. Ruth Jongejan, 302 Wolfe Street, Apt. 2, London 14, Ont.

Mr. Cal Jongsma, Trinity Christian College, Palos Heights, Ill. USA

Miss Maryke Leyenaar, RR#1, Williamsburg, Ont.

Mrs. Ann Mast, 64 Royce Avenue, Brampton, Ont.

Miss Carol J. Sluis, Group 11, Box 13, Bowmanville, Ont.

Miss Carol A. Vanderploeg, 40 Oakwood Avenue, North Haledon, N.J., USA

Mr. Nick Vandooren, 54 Gwendolen Avenue, Willowdale, Ont.

Mr. H.J. Vanhetveld, 60 Kennebec Crescent, Rexdale, Ont.

Mr. Harro Van Brummelen, 8020-160 Street, Edmonton, Alberta (coordinator)

Consultant: Dr. Arnold H. DeGraaff
Institute for Christian Studies
141 Lyndhurst Avenue
Toronto 4, Ont.

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Introduction

This report outlines some of the topics that were discussed during a two-week seminar by thirteen participants who, with one exception, were teachers of mathematics from grade 1 all the way up to college level. The survey was compiled for two reasons:

1. The seminar participants felt it would be useful for them to have an outline of the various topics discussed at the seminar; and
2. Non-participants may find it helpful in clarifying the issues we face in redesigning the mathematics curriculum for the Christian school, and in indicating the direction we must follow in writing a new curriculum.

The survey is far from exhaustive, and only a small beginning has been made. On the one hand, little work has been done in analyzing the foundations of mathematics from a Christian point of view, and, on the other, we made no attempts as yet to write actual units for use in the classroom. Our aim was to consider the basis and nature of mathematics and of mathematics learning, in order that we might construct a framework for future work in the mathematics curriculum. All of our work is of a tentative nature and the conclusions reached are <u>preliminary</u> ones.

One conclusion that was imprinted in the minds of all participants at the end of the seminar is this: our present curricula have the wrong philosophical <u>and</u> psychological basis, and a Christian curriculum cannot be a patched-up version of present ones: we must make a new start. This is a gigantic task; we hope and pray that God will give us the opportunity to continue this work in the future, and that even a larger number of teachers will join us in future workshops.

This report is a joint effort: each of the participants contributed in making the seminar a success. We want to thank especially Dr. DeGraaff for serving as our consultant on pedagogy and psychology: he contributed to this report both directly and indirectly.

H. Van Brummelen
Coordinator

CHAPTER I: THE PLACE OF MATHEMATICS IN THE CURRICULUM

1. Introduction

Knowledge cannot be neutral, and therefore it follows that a specific area of knowledge such as mathematics cannot be neutral, either. In mathematics, it is impossible to set up a system that is both complete and consistent, as Godel proved in 1931. This underscores the fact that each mathematician must hold some belief about the basis and nature of mathematics. This may sometimes be an unconscious assumption on the part of a mathematician, but it is there.

As Christians, we accept the Bible's revelation that God is the Creator of heaven and earth and that the Christian can gather knowledge about creation because God has made His creation a structured unity which God, for Christ's sake, despite the fall of man, maintains and upholds. The Bible says quite plainly and frankly that man is incapable of arriving at a knowledge of truth in its basic sense by means of a scientific theory, and if we are to start on the road toward this knowledge of truth, we must submit ourselves to God's revelation. As Paul says in I Corinthians, "God has shown that this world's wisdom is foolishness. For God in his wisdom made it impossible for men to know him by means of their own wisdom... God has made Christ to be our wisdom...each one must be careful how he builds. For God has already placed Jesus Christ as the one and only foundation, and no other foundation can be laid... For what this world considers to be wisdom is nonsense in God's sight."

Of course, Christian knowledge will not be perfect nor without principial faults. All obtained knowledge is provisional and typically human: it is incomplete and not absolute. It has to be corrected continually, and this stimulates searching for new knowledge. This is as true for mathematics as it is for other branches of knowledge. Also, non-Christian mathematicians will discover valid results, because they are results about

God's created reality. This reality cannot be destroyed by false views and interpretations.

While a difference will not be observed in all cases where a Christian and a non-Christian mathematician is at work, there are real and far-reaching differences in the way they view the basis and nature of mathematics. In the remainder of this chapter, we will try to deal with the question "What is mathematics?", discuss the relationship of mathematics with other subject areas, and mention some of the implications for the school curriculum. The discussion is from an educational rather than a philosophical perspective, and therefore we did not deal with such technical (though philosophically important) topics as a detailed analysis of the number concept.

2. What is mathematics?

Mathematics is a human activity dealing with certain ways real things function. It also refers to the results of such activity. These mathematical ways of functioning cannot be separated from other ways of functioning such as physical and biological. As a human activity, mathematics is directed by our religious position and its philosophical expression. This becomes clear in the history of mathematics and in the fact that there are various philosophical schools within mathematics.

Mathematics is a name for a conglomerate of sciences which have a few central and irreducible concepts. The two distinct aspects of mathematics are the numerical and the spatial ones. We will briefly consider the notions of number and space in turn.

The types of numbers studied in schools are as follows (in the order in which they are usually introduced):

1) Natural	N, Z^{+}_{+}	1,2,3,...
2) Whole	W, N	0,1,2,...
3) Fractional	F,(F^{o})	a/b; a,b natural (whole, b≠0)
4) Integer	I, Z^{o}	...,-2,-1,0,1,2,...
5) Rational	Q	a/b;a,b integer, b≠0
6) Real	R	??

The natural numbers are the most basic from a historical, philosophical, and pedagogical point of view. They are discrete, i.e., each one is unique and is distinguishable from another natural number. They can function in two ways: to answer the question "how many?" (cardinality) and to answer "in what position? when?" (ordinality). The natural numbers in the opposite direction leads to the integers which have both positive and negative directions. The rational numbers come about when we split a unity into parts. These numbers still answer the question "how many?" - though with different units. It doesn't take a child long to realize that he is better off getting six quarters of an orange rather than five quarters. Dividing an inch into eight parts results in each part being one eighth of an inch - a smaller unit than the inch which often is more useful in measurement. In fact, the rational numbers can be used to approximate measurements as closely as we wish, and therefore rational numbers are good enough for practical pusposes. Rational numbers also result from division and ratios.

However, when we consider real numbers things become more complex: real numbers cannot be understood apart from space and continuity. One cannot explain the meaning of real numbers in terms of natural or rational numbers: all such attempts by mathematicians have led to antinomies (paradoxes). The fact that there are numbers which are not rational was first discovered by the Pythagoreans, who showed that the hypotenuse of a right angled triangle with sides of length 1 unit was equal to $\sqrt{2}$. This number is a definite "length", and also a definite point on the number line, but yet it is not a rational number. To understand the real numbers we need the concept of continuity along a number line: using just the rational numbers there are infinitely many "gaps" in any interval on the number line, but each and every point on the number line corresponds to a real number. Thus we now have a continuous line, without any "gaps". Any one particular real number is still unique and distinguishable from other real numbers, but besides answering the question "how many", real numbers also answer "how much?" and "how far?". There always exists a real number that represents the length of any object - which was not true for the rational numbers. Thus the real numbers can be used to describe a numerical aspect of continuity (48).

The concept of continuity or space cannot be reduced to or explained in terms of number. The fundamental characteristic here is continuity or continuous extension. Though space differs from number, yet it depends on or presupposes number in at least two ways: 1) it has a number of dimensions (referring back to the natural numbers), and 2) spatial extension can be quantified by use of real numbers (distance or measure). Topology is the basic science dealing with the spatial aspect. Topology is a kind of generalized geometry in which one studies qualitative features of geometric figures that survive under transformations, like crumpling and knotting, which drastically change sizes and shapes. What we generally refer to as geometry can be seen as a sub-branch of the branch of mathematics called topology. The metric geometries including Euclidean and non-Euclidean ones as well as affine and projective geometry all deal with the spatial aspect of reality, while analytic geometry abstracts the numerical features and uses a numerical model to approximate the geometrical situation.

Thus mathematics is a composite name for different sciences such as algebra and topology. Basic mathematical concepts such as number and continuity are abstracted from physical and other everyday situations, and were developed using both intuition and man's analytical ability. Mathematics unfolded through man's activities, and it bears the imprint of human thinking. It is not infallible, nor have its precepts always been wise (1).

Mathematics arose from the needs of organized societies of people. The first concept primitive tribes recognized probably was that of a <u>discrete quantity</u>. A man needed to distinguish, for example, between having one wife or two wives, two children or three children, and soon rudimentary forms of counting were needed to communicate numbers important to the tribe. It was because they experienced numerical relations that they learned to use numbers.

At the same time, they needed some geometrical concepts: relative size, distance, direction, similarities of shapes (in order to duplicate arrowheads and implements).[2] Thus even in a primitive society, certain intuitive concepts were necessary which later developed into more systematic and structured mathematical thought. Later, in Babylonia, and in Greece, there was an enormous proliferation of mathematical necessities, and mathematics developed as a result of society's experience as well as the state of knowledge at a particular time in history.

Mathematics points beyond itself toward the other aspects of reality. Mathematics is not something in itself - if this isolation is attempted, the coherence of the universe is rent, and a mere abstraction is retained. As Hammer states: "Any attempt to separate mathematics from its applications is foolishness." [3] Mathematicians such as Nathan Court, George Polya, and John Von Neumann agree that mathematics cannot be divorced from the activities of people in society. James Newman writes that one cannot escape the conclusion that all branches (of mathematics, HVB) derive ultimately from sources within human experience. Any other view must fall back in the end on an appeal to mysticism. Furthermore, when the most abstract and 'useless' disciplines have been cultivated for a time, they are often seized upon as practical tools by other departments of science. I conceive that this is no accident"[5] *)

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*) Nor is it an accident that mathematical patterns occur again and again in nature. For example, W.W. Sawyer explains the frequency of occurence of the mathematical pattern Δ^2 as follows: "In empty space every point is as good as every other point, and every direction as good as every other direction. Laws holding in empty space may therefore be expected not to single out any particular point or direction. This considerably restricts the choice of possible laws. $\Delta^2 V=0$ expresses in symbols the law that the value of V at any point equals the average value of V on a sphere with centre at this point. This law treats all points and all directions alike, and is the simplest law that does so." [10] That $\Delta^2 V$ has applications in at least a dozen different branches of science illustrates the unity of the law structures of reality.

The pure mathematician Hardy in his famous "A Mathematician's Apology", disagrees with Newman's view: "I have never done anything'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world"(6). Ironically, his work certainly has proven to be 'useful': he established Hardy's Law which turned out to be of "central importance in the study of Rh-blood groups and the treatment of haemolytic disease of the newborn"(7), and his investigations of Riemann's Zeta function "has been used in the theory of pyrometry, that is to say the investigation of the temperatures of furnaces."(7).

In spite of Hardy's attempts to preserve the "purity" of mathematics the most important advances of pure mathematics have arisen in connection with investigations originating in the domain of natural phenomena. The late John Von Neumann argues that much of the best mathematical inspirations including that in "pure"mathematics, comes from everyday experiences or from the natural sciences. Mathematics, he says, originates with the things around us, and he shows this in detail for the development of geometry (from Euclid to Klein), for calculus (which was based on and developed for the purposes of mechanics and whose most important advances took place before its formulation was mathematically rigorous), and for the controversy about the foundations of mathematics ("something non-mathematical, somehow connected with the empirical sciences or with philosophy or both, does enter essentially and its non-empirical character could only be maintained if one assumed that philosophy...can exist independently of experience."(8)).

Von Neumann does not deny that mathematics is sometimes abstractly conceived, but then later on it will take a hand in the practical work of the world. If a mathematical discipline travels far from its empirical source, or is only indirectly inspired by ideas coming from reality, it is beset with very grave dangers; it will separate into a multitude of insignificant branches and the discipline will become, in Von Neumann's words, "a disorganized mass of details and complexities". At a great distance from its empirical source, or after much "abstract" inbreeding, the only remedy to cure degeneration is the rejuvenating return to the source - a necessary condition to conserve the freshness and vitality of mathematics (8). Undoubtedly, some parts of twentieth century "pure" mathematics will become as essential and as commonplace for the engineer of the future (not to mention other professions) as 17th century calculus has become for the engineer today (9).

In our curricula in high school, mathematics is often built up as an isolated, self-sufficient, pure body of knowledge. One of the yearbooks of the National Council of Teachers of Mathematics reflects a commonly-held view of educators in North America when it states that "mathematics itself has nothing to do with reality and can prove nothing about the world."(10) Such a statement is a declaration of faith - and it's a perspective on mathematics that determines what is taught in a curriculum and how the subject is approached by the textbook authors. My views of mathematics parallel those of Hammer and Von Neumann, rather than that expressed in the quote in this paragraph. I bring this out since the analysis of the curricula in this study will be colored by my view of mathematics.

In a nutshell: I believe that mathematics has grown from the rest of life, and that analytical reasoning has grown from experience. Mathematics must be taught from that point of view - and can be, as W.W. Sawyer has so ably demonstrated in his various books.*) Moreover, mathematics mirrors only one aspect of reality and is concerned only with the laws and properties governing the numerical and spatial aspects of reality**).

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*) See, for example, Mathematician's Delight, Vision in Elementary Mathematics, What is Calculus About, and A Concrete Approach to Abstract Algebra.

**) To borrow a geometrical example from W.W. Sawyers; We Abstract Euclidean geometry from every-day life; from "nearly triangles" and "nearly rectangles" - which, unfortunately, the texts don't mention even though those are the things we see constantly around us. If an actual rope has thickness, we neglect this in order to keep the subject reasonably simple. The position is not, as the texts seem to imply, that Euclid's straight lines represent a perfect ideal which ropes and strings strive in vain to copy; it's the other way around. Euclid's straight lines represent a rough simplified account of the complicated way in which actual ropes behave. For some purposes, this rough idea is sufficient. But for others, it's essential to remember that a rope has thickness. Thus the spatial aspect of reality is simplified in the study of geometry. Euclid's geometry does not represent absolute truth - the theory of relativity destroyed the myth, but it is still useful because it agrees with what you can see of the shapes of things - for it disagrees with other geometries only by a few millionths of an inch in every-day applications, and is a much simpler geometry to study than non-Euclidean(11).

Mathematics has an expanding area of influence, but it can represent only a small portion of human activity. Mathematics is inherently less complex than such fields of knowledge as physics, biology, common language, economics, law [12]. Mathematics differs from these fields in that it starts with the investigation and extension of much more fundamental properties of things. Mathematical thought is primarily interested in the quantitative properties of collections and in the spatial aspect of the universe.

Yet mathematics and mathematical models have an important function in the development of other fields of knowledge. Because of the intensive development of certain concepts, mathematics has often been considered just as an object language for the physical sciences. At the same time, logicism has deified logic and made it the foundation, origin, and goal of the cosmos, and believes that mathematics is nothing but a part of logic. In both cases, mathematics has been hemmed in and the security sought - and often achieved - has its price in the applicability of mathematics to any but comparatively simple situations. Mathematics has had amazing successes and yet remains, in its present state, applicable to principally simple problems [13]. It is to be hoped that the publication of The Mathematical Sciences, A Collection of Essays [14] by a group of eminent mathematicians is an indication that mathematics is not seen any more as just a part of logic, and that it has an important role to fill in areas other than the physical sciences. I will expand on this in the next two sections.

3. The Relationship Between Mathematics and Logic

We must not make the error of equating mathematics and logic. Rational analysis is not limited to mathematics; it is also found in naive experience, and proof plays a role in all sciences as well as in practical life. Intuition based on experience "must retain its complement of logic." [15] Logic is a useful instrument of demonstration - and an essential one - but at the same time it is an insufficient one, particularly in the discovery of new mathematical methods.

In the development of the various branches of mathematics, clear insight into the structure of a branch was obtained only after a long period of trial and error, and "it seldom occurs that mathematicians are significantly aided in-their struggle by the results of formal logic, as also Beth confirms."(16) Kuyk explains the difference between mathematical and logical thought as follows:

> 'Thus there is room to conjecture that to mathematics there corresponds a way of thinking which differs from the ways of logic, in that mathematical thought (initially) concentrates on different sort of things (e.g. the basic "material" of numbers and geometric figures, etc.) and on that, which by way of abstraction and construction, can be made from those things. Logic, on the other hand, directs its attention towards the logical questions which lie at the root of all human thinking. Hence, one may say that (mathematical and formal) logic is in some sense secondary to mathematics, as it is indeed on thought-through and worked-out disciplines, which may serve as its testing fields. We may put the relation between mathematics and logic in another way. It is the mathematician who goes a long way to develop and explore a certain field by expounding its main features and determine its principal theorems, whereas the logician may try to axiomize the discipline further rigorously by re-exploring it on its logical merits ... For the logician the living content of a mathematical theory is not as much the results of the theory as the syllogistic structure of it."(16)

In contrast with this view, logicism reduces all aspects of reality to that of analytical thought. It holds that mathematics must be based on a small number of axioms which express "simple truths" of logic. Logicism has achieved great things. However, reductionist philosophies such as logicism and formalism have also led to the crisis in the foundation of mathematics; today it is well known that logical difficulties such as paradoxes and contradictions do not occur only in the humanities, and what is more, it has been shown that these paradoxes cannot be solved using logic alone.

Bertrand Russell - together with Whitehead - probably was the most famous mathematician of the logistic movement. He tried to reduce the concept of number to that of sets (which he calls classes). Russell admits that the logical addition of 1 and 1, according to the principles of symbolic logic, would always yield one as its result. That's why he gives the following definition: "1 + 1 is the number of a class -w- which is the logical sum of two classes -u- and -v- which have no common term and have each only one term."(17) However, the antinomy Russell tries to avoid by introducing the class-concept, reappears in the vicious circle of his definition: he tries to deduce the concept of number from the concept of class, but for the simple distinction of the classes he needs number in its original meaning quantity: the number two is defined by using the number two (18) Russell claimed that the introduction of his so-called theory of types avoided such apparent circularities and was inherently reasonable, but Barker describes how at best it is a makeshift device with many unattractive consequences[19] *). Despite Russell's failure, many mathematics courses imply that the concept of set is a more basic one than that of number, whereas it is the concept of discrete number which is the essential "stuff" at the basis of arithmetic.

The essential insufficiency of formal logic was made clear in a particularly striking way by Gödel's theorem on the occurrence of undecidable propositions in formalized mathematical theories, a theorem which implies that formal logic is incapable of ever containing the whole of intuitive mathematics[20]. As Kneebone says:

> "Exclusive reliance on formal logic would in fact necessitate complete formalization of the axioms on which mathematics is based; and Godel has proved that, whatever systems of axioms may be adopted, there will always be propositions which can be stated in a language of the system and even decided by intuitive means, but which nevertheless are logically inaccessible from the chosen starting point. Thus, indispensable though formal logic is to mathematics, it cannot provide an ultimate criterion of validity of mathematical assertions."[21]

Thus mathematics is not exclusively a logical, deductive science. Attempts to suggest that mathematics is part of a safe, secure, logical structure existing independently of human experience are erroneous. Theorems almost always originate inductively and experimentally; only after they have been accepted (and used) are they proved deductively.

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*) Besides logicism, there are two other distinct streams of mathematical thought. Intuitionism holds that logic is a part of mathematics, rejecting such logical methods as indirect proof. Something exists only if you can construct it with your mind; "constructibility" is the criterion for truth: certainty is obtained if each step in the construction is intuitively acceptable. Intuitionism places man and his thought in the foreground and loses sight that man's knowledge is incomplete and imperfect. Intuitionism unnecessarily restricts the methods and results of mathematics, and has not influenced school curricula in North America. Formalism tries to fuse axiomatic and logicistic methods, reducing mathematics to a string of formulas or assertions using mathematical and logical symbols. You cannot talk about the truth of adduced results but only about the validity on the basis of the assumptions. Formalism attempts to separate mathematics from reality; its theorems are not about some phase of the existing world but about whatever is postulated by thought. Thus mathematics is true simply by definition or convention and not because it (correctly) theorizes about reality. It is logicism and formalism which by and large have determined the mathematics curricula in North America today.

Bruce Meserve states this as follows: "In most, if not all, elementary mathematical systems the axioms (postulates) originated as abstractions from observed properties of the physical universe"[(39)]. Kooi adds that "an abstract terminology has the advantage that one is not bound to a particular interpretation of the system...but during the development of a logical system, mathematics usually keeps in mind at least one interpretation, in order to be certain that the system will lead to results that have something to do with reality. The degree to which the mathematician is bound to reality is usually greater than he is willing to admit"[(40)].

Strydom [(41)] gives several examples: 1) the formal, axiomatic definition of measure is abstracted from the physical properties of length, area, and volume; 2) negative numbers were first introduced by the Hindus to describe "debt"; and 3) non-Euclidean geometry and many-dimensional geometries were developed by questioning Euclid's axioms, but the results were mistrusted because they disagreed with the naive experience. They became generally accepted only after the results were shown to have physical interpretations (e.g., geometry on a sphere, theory of relativity). To quote Kooi again: "That mathematics keeps itself at a distance from reality usually is a deceptive appearance. It can free itself at any given moment from what is generally considered as reality and this can give opportunities to develop new, undiscovered fields. One could compare mathematics with Columbus, who "distantiated" himself from the then-known "reality" and discovered the unknown America" [(40)]

Even the "theoretical" Greeks drew on experience: Pythagoras obtained his proof (of many possible ones) by trial and error, limited by his experience and knowledge, and trying to prove the theorem only after his experience had convinced him of the truth of the theorem. In a mathematics class it is wrong to imply that Pythagoras dashed off his theorem just like the book dashes it off. Similarly, Euclid's Elements gives a summary of Euclid's achievements, but, as Hammer points out, the Elements do not reveal how to do mathematics; it only gives a form of presenting it after it is done [(22)]. Hammer also shows that Euclid, though one of the most brilliant mathematicians in our history, had detrimental effects on the development of mathematics:

> "The treatment of Euclid's geometry as a model of reasoning is one of the reasons for the slow development of mathematics. In effect, it has been used largely to prevent reasoning, by its use as an authority. The geometry taught as a model of thinking has actually been used as a mental strait-jacket. A heritage can be a curse as well as a blessing." (23)

Euclid is a prefabricated house, and its construction is static. It cannot be made dynamic by giving our pupils a systematically ordered catalogue of tasks to accomplish, which is what we do in teaching Euclid[24]. Instead, we need problems that read: "Here is a situation - think about it - what can you say?"[25]

An over-emphasis on logic and proof - as found in many of our modern mathematics curricula - causes mathematics to become rigid, severe it from reality, and make it a purely logical game. A formal logical approach to mathematics in school (usually one with heavy technical detail) tends to smother the student's intuition and imagination, and he will become so rigid that he can do nothing in a typical applied situation.

Rather than implying that mathematics is nothing but a logical exercise of the human mind, we must teach our students that mathematics enables us to understand some of the law structures of the world around us. A logistic approach to mathematics contributes to an alienation from the view that the universe is a structured unity. In a broader perspective, Lanczos claims that this approach will augment the tensions of our time, since as a result we lose sight of the eternal and inviolable laws to which we all are of necessity submitted, we start to believe that we can control the world and its development, and in our egocentredness constantly increase our demands.[26] By emphasizing the pure logic used in mathematics, the human mind withdraws into itself and loses touch with the fundamental substratum that everything is subject to the cosmic law order.

I do not deny that logic and proof have an essential place in mathematics, and that there should be a place for them in our mathematics curricula. But proof must be shown to be a <u>tool</u>, and not - as usually has been done in geometry - the essence of the subject. Moreover, a teacher should not require a student always to give back the same proof as presented in class - this puts a penalty on analytic thought. To show that logic is a useful tool, students should participate by trying to construct counterexamples, detecting errors in false statements, trying to formulate proofs themselves, or improving on the theorem or its proof.[25] A student must be led to understand the difference between that which they admit and that which they prove between the termination of intuitive methods and the use of deductive reason, and between the physical object and its representation by a drawing and the abstracted concept used in formal deduction. At the same time, a student must recognize that logic is only one of the tools of mathematicians, and that its use is not limited to mathematics. He must be shown that while either inductive or deductive properties may come to the fore, they remain aspects of the total unified structure of knowledge.

In conclusion, "to present mathematics entirely in the rigorous deductive spirit not only precludes any possibility of applying mathematics, it is dishonest, even as a picture of contemporary pure mathematics."(27) An intuitive approach to new topics with many different intuitive considerations is sound both from a philosophical and psychological point of view. In our teaching we must make clear that our every-day integral experience, whose meaning is guaranteed by the creation order, is the foundation of all scholarly work - including that of mathematics.

4. The Relationship of Mathematics to Other Areas of Knowledge

It is not possible to treat adequately the place of mathematics in our schools without boking at its relation with science and technology - and, indeed, with all other branches of knowledge. Of course, modern physics is inextricably bound up with mathematics, indeed, physics expresses all its findings mathematically (e.g., the mathematical aspect of kinematic motion is represented by a differential equation, and hermitian operators are used as mathematical approximations of physical operations in quantum mechanics). What is more surprising is that today also the chemist, the biologist, the psychologist, the linguist, the social scientist, the economist, and the political scientist more and more are using mathematical models in trying to solve problems that they meet in their fields. The importance of mathematics lies in its applicability to these other fields, and in the fact that the same mathematical structure can serve as a model for many seemingly unrelated problems.(28) This means that we need to teach mathematics of the modern structural type, where we stress the unifying ideas and principles that we meet in both the internal and external applications of mathematics.

We must include applications of mathematics to other fields in our curricula. We should be sure that the experimental background and the mathematical identifications of the model must be in the student's experience(29). If this is done, a student not only deepens his understanding of the mathematical concepts and techniques, but his studies in mathematics become more relevant, leading to better motivation.

At the same time, it should be pointed out to the student that the use of mathematical models in describing the real world is a very delicate matter; the real world is far more complex than any mathematical model. In an application, we are abstracting only the <u>mathematical</u> aspect of the situation and incorporating this in our mathematical model. The simple process of a housewife attempting to solve the problem of packing as many dishes as possible in a box is more difficult than has been solved in the far reaches of measure theory;[30] and when a psychologist tries to help a mentally disturbed child, he has a problem that is far too complex for mathematics to be of help; the mathematical aspect of such a situation is only a minor one.

The question arises why mathematics has been applied fruitfully in physics, and, to a somewhat lesser extent, in chemistry, but not in a field such as biology until very recently*). One reason for this is that in the hierarchy of the various sciences, biology is a more complex one than physics and that therefore the mathematical aspect of biology is more difficult to extract from a biological situation. Dooyeweerd gives another reason: mathematics, he claims, has been walled in and imprisoned by the absolutization and logicistic reduction of the mechanical and logical aspects of mathematics and physics.[31] He continues that biology should realize that physical methods of inquiry can only be sufficient for the investigation of the physical substratum of the organic-biotic aspect of reality. It will then with increasing emphasis insist on the desirability of a mathematics of specifically biological orientation. Dooyeweerd has proved to be a prophet; he first published this statement in 1938, and today a new field of mathematics called biomathematics is beginning to emerge. So far, the mathematical techniques have been conventional, but there is good reason to expect that within the near future biology will inspire the development of new mathematics.[32]

Similar observations can be made about other fields. Mathematical concepts such as mappings between sets, partially ordered sets, and semi-groups are being applied in analyzing the basic structural properties of languages in the new science of mathematical linguistics now flourishing in the U.S. and the U.S.S.R.[33]

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*) It should be noted that modern physics has had to create the new mathematics it required on its own account, as it needed it in its development; mathematics, for example, was unable to provide quantum mechanics with an adequate concept of space.

Also, some of the non-statistical techniques that social scientists find most useful have been developed in recent years and in the future their problems will likely inspire much new mathematics. Mathematically speaking, the social sciences will be far more difficult than physics; the negotiations of a committee are far more complex than the orbits of a solar system.[34] The techniques of linear programming, graph theory, differential equations, and computer simulation have been used effectively as tools in providing mathematical analogies of social science situations, and in solving some of the mathematical aspects of such situations.

A one-sided mechanistic and logical orientation also has prevented pure economics from analyzing the complex structure of mathematical analogies in economics, and this has caused tension between the "laws" of pure theory and the factual side of economics[35]. Only recently has such an analysis been started, and it will likely call forth the development of its own branch of mathematics[36] Hopefully economists will at the same time remember that economics cannot be <u>reduced</u> to mathematics, and economists will always have to master much more than just the mathematical aspect of the subject. The same is true for such areas as art, and music. That mathematics plays a basic role in both music and art has been discussed in many articles [37], yet everyone realizes that neither art nor music is a part of mathematics.

5. The Use of the History of Mathematics in the Classroom

Generally speaking, we do not follow the same path in the teaching of mathematics that was followed by mathematicians when they first developed the concepts. However, we should not lose sight of the way in which mathematics developed. Glaisher asserts that he is "sure that no subject loses more than mathematics by any attempts to dissociate it from its history"(42). The history of mathematics can increase the understanding of mathematics, show the student how and why it developed and changed, indicate its relation to the other sciences, and help the student understand the nature and structure of mathematics.

Some of these things are discussed in more detail in a recent yearbook of the N.C.T.M., Historical Topics for the Mathematics Classroom. While a book of this type is useful, a little history "added on" by a teacher or text at certain points in the text course is not as useful as certain historical developments being made an integral part of the curriculum. This means more than giving a few biographical notes here and there; we must present historically important problems and ideas that will help students "to perceive relationships and structure in what appears to be a tangled web of geometry, algebra, number theory, functions, finite differences, and empirical formulas"(43).

Our students at the moment lack a historical perspective. While the historical approach is not always the best way to communicate insights, it can often help tremendously. A historical approach would give the students some indication of how mathematics has been influenced by various civilizations, and how in turn mathematics has influenced Western culture. Our students need a historical perspective to be able to understand today's society. Unfortunately, I am not aware of any curriculum that has consistently tried to implant such a perspective in their courses. An effective program would require much careful planning and a great deal of research to find topics that are (1) mathematically significant, (2) historically significant, and (3) pedagogically feasible.

One such topic could be non-Euclidean geometry, whose development could start with Euclid and the Greeks, continue with tracing the history of perspective in art and the parallel development of projective geometry, and conclude with the study of modern non-Euclidean geometry and its present-day applications.

a) <u>The Focus of Teaching the History of Mathematics</u>
The study of the development of mathematics proper as it is embedded in a broader cultural and scientific context. The approach should be two-directional: the cultural context as it directs and gives meaning to mathematics, and mathematics as it influences developments in the rest of culture.

b) <u>Considerations to Keep in Mind</u>
Care must be taken to see the web of relationships in which the development of mathematics is entangled. The important associations of any phase of the development ought to be deciphered and presented as influences integral to the development of mathematics proper. Mathematics has not developed in isolation and should not be treated separately.

Care should be taken to place emphasis upon the sweeping trends and underlying motives working in the development of mathematics. A study of the history of mathematics should not be a mere cataloguing of disjointed facts as if mathematics develops by accumulation of new facts. No science develops in this manner. New ways of posing old problems or seeing old facts often play a key role in bringing about a turning point.

Care should be taken to keep an over-all picture of the development of mathematics in mind while studying a small segment of mathematical history. Then from our present vantage point we should be able to see omissions and oddities in past mathematics that will tip us off as to the general character of that phase of development. The outlook and limitations of a certain period of mathematics can often be detected by noticing which things are peculiar to that phase or different from earlier or later phases. Also, the present can be understood only in terms of the past.

Care should be taken to deal honestly with the development of mathematics. Discoveries should be viewed in their own historical context of meaning. Later day meanings and tendencies may not be read back into those discoveries where they are not present.

c) <u>Proposed Interconnection</u>

Prime considerations in teaching mathematics ought to be pedagogical soundness and systematic coherence. These norms can be met by using the history of mathematics as a tool for introducing and explaining the various topics in mathematics. The use of the history of mathematics must be integral to the structure of the mathematics program. Historical considerations should not be appended just for interest's sake. Biographical sketches or interesting anecdotes may please those students already curious about how mathematics developed, but they do not give students the realization that the subject matter of mathematics has a history. The history of mathematics should be used whenever it is relevant to the topic under discussion, and probably not otherwise. The historical approach to a topic is not necessarily the best one at all times, nor is the developmental, chronological order always the best one to use from a pedagogical point of view.

d) <u>Justification for such a Use of the History of Mathematics</u>

Using the history of mathematics in teaching mathematics should produce the delightful realization in the student that mathematics is not a static body of knowledge. This approach will show mathematics to be a human and sometimes erring attempt to formulate how numerical and spatial functioning are regulated.

It should reveal that mathematics proceeds from the concrete to the abstract. Mathematicians proceed intuitively at first, only later are formal connections made between the results. This should show that mathematics is not a formal process of deduction of results from for-all-times-accepted axioms. It should show that mathematical discoveries are often stimulated directly by concrete problems met up with in real life.

It should show the student that mathematics is connected with many other areas of human activity. More particularly, it should show that the development of mathematics was limited and directed by a general view of the world and mathematics' place within it.

Studying the development of a certain topic may give the teacher some beneficial insight into how to teach the topic pedagogically. Various approaches to the topic may be unearthed for the teacher to evaluate. Perhaps the historical refinement of a concept would be the best approach to introducing that concept. Or maybe the history of a topic will reveal various pitfalls that students will also be prone to fall into. At any rate, a historical exposition of the development of a topic will give the student more understanding conceptually than a brief, formal definition which he cannot relate to. In some instances, the teacher may be able to use profitably his knowledge of classical problems which lead to a development of some area of mathematics.

6. Some Implications for the Curriculum

It is not my intent at this time to give a detailed catalogue of what mathematics should be taught, nor how the topics should be approached in a high school curriculum. That is beyond the scope of this study, as well as beyond the scope of just one person. But certain implications do follow from what I have said in this chapter:

a) Mathematics must not be taught as something existing for and in itself; applications must be shown, and the curriculum should bring out the place of mathematics in the development of human culture. Almost all major fields of human endeavour and innumerable situations in everyday life, are likely to lead to significant applications of mathematics. We must find problems which are complicated enough to represent a real situation honestly, but simple enough so that students have some chance to solve it. This is not an easy task - as the large number of artificial and insignificant problems in most texts indicate. We must also remember which fields will likely have to be of major importance in the future as far as applications are concerned; classical analysis, linear algebra, probability and statistics, and computer science[(44)]. Also, we give a dishonest picture of mathematics if we do not allow the student to participate in finding the right problem or theorem from time to time. These applications do not necessarily have to be deep or remote. Interesting applications of mathematics can be found in many everyday situations. As Pollak adds, "Not every problem that we attack will turn out to be easily solvable, but the potential for interesting situations is all around us."[(45)]

b) The unity of the structure of the cosmos should become evident to the students; more emphasis will have to be placed on mathematics as the study of structures. Not only has the structural approach given mathematics its power and its ability to consolidate and simplify the diverse mathematical theories that have evolved during the last two hundred years, but an understanding of this approach will also train students in mathematical ways of thought. We must not confuse a structural approach with a logicistic approach or formalistic approach; mathematics does not start with the finished theorem; it starts from situations. Before the first results are achieved there must be a period of discovery, creation, error, discarding and accepting. Modern mathematics is not to be equated with the axiomatic approach: I believe that the structure of mathematics can only be understood after a long, initial, quasi-experimental investigation. The way in which Euclidean geometry is presented in most North American highschools very seldom leads to an understanding of the structure of geometry and even less seldom to a correct view of the place of logic and proof in mathematics. From a structural point of view, teaching geometry using the transformation approach is much superior: it calls for the use of matrices, and the matrices can then in turn be used to develop the theory of transformations. This can also lead to the development of the addition formulas for the sine and cosine functions in trigonometry (see, for example, P. 226 of (38)), the properties of complex numbers, and the investigationof vectors. In each of these internal applications, applications in other fields can be demonstrated relatively easily, so that the teaching of mathematics can become better integrated with contemporary applications in industry and research, and, at the same time, be able to evoke a mathematical response from the pupil(46).

c) The content of the school mathematics course should be modified by the introduction of material that has become significant, either because it is comparatively new, or because it is much older but now has a significance which it lacked before. Certain topics have become of central importance in various fields in the last fifty years, and they must receive more emphasis than has been the case in mathematics curricula in Canada. Examples include: transformations and vectors;

probability and statistics; matrices, fields, and groups; calculus and analysis. This means that the time spent on other topics must be reduced; deductive geometry, simplification and factoring of polynomials as a topic in itself (after students are introduced to the principles, most of the practice should occur in problems in other chapters where such simplification is needed), and compass-and-straightedge constructions in geometry. In our choice of subject matter, we must try to show and use the underlying unity to strengthen understanding.

In conclusion, to show that there are viable alternatives to the formalistic approach used by the majority of texts that are offshoots from the UICSM or the SMSG projects, I point to the School Mathematics Project texts published in England, and summarize some of Dieudonne's ideas about the teaching of geometry:

1. Nobody need be concerned, in secondary schools at least, with teaching the future professional mathematicians (not to speak of the great ones), of which there may be one in 10,000 children. What is really at stake is the kind of mental picture of mathematics that will emerge in the mind of an average intelligent student after he has been subjected to that treatment for several years.

2. A mathematical theory can only be developed axiomatically in a fruitful way when the student has already acquired some familiarity with the corresponding material - a familiarity gained by working long enough with it on a kind of experimental, or semi-experimental basis, i.e., with constant appeal to intuition.

3. When logical inference is introduced in some mathematical question it should always be presented with absolute honesty, - that is, without trying to hide gaps or flaws in the argument.

4. Geometry should put the emphasis not on some artificial playthings as triangles, but on the basic notions such as symmetries, translations, composition of transformations, etc.

5. Whenever possible, any notion should be developed both from the algebraic and the geometric point of view. Throughout geometry at high school level, the emphasis should be on linear transformations, their various types and the groups they form.

6. The curriculum at high school level should deal only with mathematical objects that have an immediate intuitive "interpretation" of some kind.(47)

Everything that I have said thus far deals only with the content of the curriculum. Before the writing of any part of the curriculum, and, indeed, before even the formulation of specific objectives for a mathematics curriculum, the nature of education and the psychology of mathematics learning will also have to be considered.

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CHAPTER II

THE LEARNING OF MATHEMATICS

1. What is Education?

The views of education given in this section are described in more detail in DeGraaff's essays on "The Nature of Education" and in "The Nature and Aim of Christian Education".

Man has been created by God in such a way that he can hear and respond to God's Word. God speaks to the heart of man, and man cannot escape this powerful Word of God (cf. Romans 1). Man's entire life will be a response that is a service of either God or of an idol. Thus teaching and learning are of a religious nature and are done either unto God or some pretended god. Both teaching and learning are normed and responsible activities.

Human development is not an automatic, natural process, but requires pedagogical influence and interaction, and the exertion of formative power. Education always implies a deliberate attempt on the part of the educator to lead the child or the adult in a particular direction according to certain norms.

In our forming we are bound to the nature of our "object". A human being never functions as an "object" in the same sense that an animal would function when being "drilled" in a certain task. For example, only man can experience the feeling of satisfaction and joy after having solved a difficult theoretical problem; only man can appreciate the beauty and structure of art or music or mathematics. Unlike trainers of animals, educators of human beings, must appeal to the personal responsibility of the person being educated; the person must be actively involved in the educational process.

Education requires a fundamental respect for those we seek to educate, because they are human beings made in the image of God, created to respond to His calling. A child does not develop into a person, he is a person from the start, though an immature one. Already at a very early stage the child begins to assert himself and to take an active part in the educational process. As educators we are only directing ourselves to the various aspects of a person's existence to prepare him for his calling in life, but all the time the person remains a free human subject, who is given emotional freedom in a climate where his opinions and reactions are respected even if not approved. He must have the courage to tackle the tasks at hand. To educate means to give actual direction to the development of a person's life, to lead him towards a particular goal according to certain norms, to unfold and develop him; in short, to educate means to exert a real formative power.

Education requires liveliness, inspiration, stimulation, care, and genuine concern for a person's development. However, when man's nature (his freedom and responsibility) is not respected, education invariably turns into a pure demonstration of power and domination, or it becomes mental persecution, manipulation, mechanical training, or it is reduced to over-protection, doting, or a mere laissez-faire attitude. Educating is fundamentally different from forcing someone to submit to one's will or from "mental engineering". The violation of human nature inevitably leads to pedagogical impotence and failure.

In short, both teaching and learning are normative and responsible activities, for which a person is responsible to God. The pedagogical desire to form on the part of the educator ought to demonstrate itself in his respect for and appeal to the religious selfhood of the one being formed, and the pupil ought to have the freedom and the responsibility to take the guidance that is given to heart. Mathematics must be taught so that the student is shown how mathematics helps him to fulfil his calling in life, and must enable the student to be a full, responsible human being who is actively involved in the educational process: the student must participate, co-operate, and be given the opportunity to initiate.

2. <u>The Objectives of Teaching Mathematics in the Christian School</u>

At the outset it is necessary to briefly summarize the nature and aim of education in order to formulate objectives of teaching mathematics that are embedded in a larger framework. We have tried to posit these objectives so that they are consistent with this framework.

The aim of education is to direct each individual so that in his own unique way he may learn to serve God according to His Word. In this way the student grows in ability and willingness to employ all his God-given talents to the honour of God and for the well-being of his fellow creatures, in whatever area of life he is placed by God. (<u>NOTE</u>: The surrender of the individual to God's will is the work of the Holy Spirit who uses us as His instruments).

The purpose of the school is that the student learns to discern truthfully in order to gain a deeper understanding of his many-sided (religious) tasks in life.

Our understanding of the creation determines our perspective. For example, if we view it as a 'playground', the creation becomes something to manipulate and exploit as is being done today. However, if we view it as a totality with many diverse sides to it which harmoniously balance and compliment each other and adhere in Jesus Christ, then we necessarily have a different perspective. The Cultural Mandate calls us to responsibly preserve and develop God's creation. (There is more to man's responsibility than just that but conveniently we will not go into that now). Mankind is called to have dominion over God's creation.

We must therefore recognize the rightful place of mathematics as a science which investigates in detail and describes one of the aspects of the universe about us. This is not to say that we can ever achieve full comprehension of all the things about us but that we use math as a functional tool to develop and preserve the creation which is to the upbuilding of God's Kingdom. Thus, we should not and may not fragmentize our efforts and pursue areas as an end in themselves. Nor should we be concerned about producing future mathematicians at the elementary and secondary school level so that we gear the curriculum accordingly. We should always seek to contribute to a better management of creation with God's Word as our guide. As Taylor says:

> The Bible does not teach about the facts of science but it does provide us with the ordering principle in terms of which the data of science may be understood. The Word of God indicates to us the why of our creation, not the how. It provides us with the indispensable background and sense of purpose of this mysterious universe and of our own position, role and destination within it.

THE SPECIFIC AIMS OF MATHEMATICS IN THE CURRICULUM ARE:

a) The student should gain a better understanding of the concepts of number and space so that they can be abstracted from concrete situations, that they can be theorized about, and that the results of such theorizing can be applied to concrete situations.

The latter could be both physical and non-physical examples such as water and Gross national product respectively. Only in light of investigating all sides of these examples can we fully appreciate their meaning and relation to reality. The mathematical aspect is but one side which is integrally related to the various other aspects. Although this may seem intuitively obvious, gross misconceptions have their beginning at this point. Thus, many a student and/or teacher often refers to water as H_2O and means to imply thereby that it is the complete reflection or picture of water. There are many more ways in which water can be studied than from a chemist's viewpoint. To do otherwise would be to absolutize a relative aspect. It may be useful at this point to note that a certain amount of abstraction is necessary in every subject. "Mathematical models" then, in physics for example, should be seen as an attempt to clarify a physical object by abstracting its mathematical aspect.

b) The student should come to realize that math should serve as a functional tool in solving our everyday problems and as a handmaiden for the explanation of the quantitative situations in other subjects. To illustrate the point, for the former the elementary basic notions of measurement (volume, temperature, etc.) are necessary for baking a cake, while for the latter more sophisticated formulas such as Einstein's relation [$E = MC^2$] may be necessary.

c) The student should come to realize that math is a developing science, and that throughout history it has influenced and in turn is influenced by cultural forces. For example, one of the over-riding cultural forces, viz., prestige and world recognition, has driven American society to stress heavily math education and research in the post-Sputnik era. Conversely, the more sophisticated technology which has resulted from this emphasis has contributed to the manipulation and fragmentation of American society. This is partly responsible for the alienation of youth from the American "way of life", but such alienation will probably not contribute to a better understanding of the rightful place of mathematics.

If the curriculum is structured so that it meets these objectives, it should follow that the student gains respect for the laws which hold for God's creation order. Mankind is bound to these laws and must deal with them accordingly. He must treat goldfish, for example, as aquatic creatures and not as terrestrial ones. There is a certain "universal validity" to these laws which are dependable and can be trusted deeply. Gravity on earth never fails us. And even though we may make mistakes and errors in our theorizing, reality is harmoniously and integrally "consistent". Keeping this in mind, the student should become aware of the power as well as the limitations of mathematics. Try as he may, good intentions (or bad ones for that matter) will not change the law structure of creation. The student should be led to trust the dependability of God upholding these law structures.

Also, mathematics should provide an opportunity for the student to discover the order, patterns and relationships that exist in creation. These ideas are some of the boxes in the skeleton of mathematics. By having a better understanding of these ideas the student may appreciate more fully the aesthetic features of mathematics. Puzzles and games may also serve useful to that end.

The realization of our objectives may be helped by the development of the student's techniques and skills. The degree of facilitation should be individualized in relation to the student's abilities and direction with respect to his many-sided calling in life.

3. Some Psychological Considerations in the Development of a Mathematics Curriculum

(a) Discovery and Expository Learning

Ausubel makes two distinctions in the types of learning taking place in the classroom: i) rote learning vs. meaningful learning, and ii) discovery vs. expository or reception learning. Learning is certainly not either the one or the other in each of these cases, but may be viewed as a continuum.

Rote learning processes are relatable to cognitive structure in an arbitrary, verbatim fashion. There's no anchorage to previously established ideas and concepts, no connections, and therefore retention span is short. In meaningful learning, on the other hand, new ideas are related to relevant, established ideas and therefore there is more retention and better understanding.

Reception and discovery learning can each be rote or meaningful, depending on the conditions under which learning occurs. Expository learning is often abused, but is efficient, leading to sounder and less trivial knowledge than when pupils serve completely as their own pedagogues, especially when the student has reached the stage of abstract and formal operations. There should be a gradual change from a structured discovery approach with concrete materials in lower elementary levels to an approach that is mainly expository at the end of high school.

If expository learning takes place in a meaningful manner, it is as active as discovery. The student must judge the relevance of the new concept, and blend it into his personal frame of reference. Dangers of reception learning are that the learner may delude himself into believing that he has really grasped precise meanings when he has grasped only a vague and confused set of empty verbalisms, and that the student may not be motivated for active, meaningful learning and yet in class give the appearance of being "with it". For active meaningful learning, Ausubel claims that we must (i) teach central, unifying concepts first, (ii) observe the limiting conditions of developmental readiness, (iii) stress precise definitions (but research shows that some ambiguity and equivocation in the use of certain terms is not particularly detrimental to learning), (iv) stress similarities and differences between related concepts, (v) make learners reformulate new propositions in their own words so that they have real meaning for him in terms of his own structure of knowledge, and (vii) question for pseudo-understanding.

We must capitalize on the availability in the students' cognitive structure of relevant anchoring ideas reflective of prior incidental experience. For example, we should proceed from intuitively familiar to intuitively unfamiliar topics in sequencing subject matter (although our theorizing students can visualize), and arrange a subject matter field as far as possible in accordance with natural sequential dependencies among its component divisions, so that the students will come to see the unified structure of the subject.

Evidence of meaningful learning includes (whether this be of the "discovery" or of the "expository" type): independent problem solving (but the inability to solve certain problems does not necessarily mean a lack of understanding), and presenting the learner with a new, sequentially dependent learning experience that cannot be mastered without a genuine understanding of the prior learning experience. Meaningful learning is easier to learn and remember, more rapid, remembered better, and there is less interference than in rote learning. At the same time, it must be recognized that there are some situations where there is some necessity for rote learning (e.g., memorizing one's phone number).

As far as learning subject matter and specific skills is concerned, studies on the use of "discovery" methods of learning under controlled conditions do not report positive findings. However, these reports are to be questioned: (1) the "discovery" in most cases is of the abstract type (e.g., drawing conclusions from certain number patterns), while the type of discovery we would advocate would deal with the discovery of mathematical properties from concrete materials found in the classroom (see the next section on a math-lab approach), (2) it is likely that concrete discovery techniques are beneficial in long-term learning only, and most present experiments are based on short-term research, and (3) most research studies test the students for specific"skills", and not for the students' appreciation of the nature of mathematics, its relation to other sciences, and its place in our culture. Also, we feel that it is likely that students that have learned by proper discovery methods will catch up with the written computational processes of the traditional procedure after a certain amount of time. At about age 11 or 12, the student will start to systematize his knowledge.

We do recognize that there is some research indicating that certain personality factors of the individual learner, including submissiveness and conceptual level, are related to the effectiveness of discovery methods. Therefore in a math-lab approach we would not exclude the availability of more standard "workbooks", so that students preferring this approach may use them. Different approaches may well be effective with different specific mathematical topics and with different students, and we should ask ourselves: "What kind of discovery, with what kind of materials, with what kind of learner,

(b) Individualizing Instruction: the Math-Lab Approach

Almost every educator agrees that an ideal we strive for in the classroom is to individualize the curriculum for each learner, so that each child may develop his talents in the best possible way. How far this goal is possible to attain remains to be seen, but the approach we envision is a far cry from the static, everybody-turn-to-page-thirty approach now used in most schools.

At the secondary level, it is suggested that units be written so that each student can arrive at a thorough understanding of the basic concepts (perhaps introduced in a physical or historical setting), and that the majority of students would go further to a level which relates to their own abilities and directions.
It would be necessary to grasp the basics in Unit 3, for example, in order to understand the basic in Unit 4, but students could go to different levels in different units. Even a student who studies only the "basics" in each unit in high school should at the end of his high school studies have a grasp of the three main objectives that we listed in an earlier section, although not at the depth of understanding that other students would. There would be fewer discovery experiences at the high school level, but units should always be introduced using concrete situations that the student can visualize, and the results of the mathematical analysis should eventually be shown to help in understanding concrete situations.

At the elementary and, to a lesser extent, the junior high level, a math-lab approach should help to make the learning of mathematics more meaningful. The aims and goals of authors of present mathematics textbooks usually seem very noble, and very much in line with what we mean by the learning process:

> "As in previous books of this series, emphasis is placed on understanding the general principles and processes of mathematics. To give insight, meaning, and understanding a place of first importance, the discovery approach has been extended in this text. The object of this book is to stress both the understanding and logical development of each topic and the acquisition of skill in the application of the

> "This book was written for you, no matter what mathematics you may use in the future. In it you will be helped to learn what mathematics is about; how we use mathematical ideas to solve simple and complicated problems. It is important that, as you work through this book, that you think for yourself. Mathematics is not a subject in which we memorize many answers; it is a subject in which we learn to think in a certain way". (from Contemporary Mathematics - Holt, Rinehart, Winston).

What happens then? Usually the following: The textbook falls into a certain pattern of lengthy (often useless) discussions, and long tedious exercises. The teacher usually embraces this textbook as his bible and goes through it in a page by page analysis, struggling with the slow as well as with the gifted, to keep them at the same page in the book. This continuous repetition, without much originality, aside from stifling thought and destroying enthusiasm, causes 1/4 of the class to excell, 1/2 to plod along, and 1/4 to be constantly in the "doghouse". It is safe to say that at least 80% of all math programs fall into this category.

The aim of a math-lab approach is to make use of the natural experiences of children in order to develop the basic ideas of mathematics. In it, the teacher observes, plans, directs, and checks for progress while the children experiment.

The advantages of a math-lab approach include:

1. The emphasis is placed on true child discovery. It is a proven fact that a child learns better when a meaningful method has been provided to clarify a certain skill. All too often we teach methods before the children have realized the need for these, and they are usually provided by the teacher on the blackboard.
2. A math-lab approach "streams" the pupils according to their talents. The fact that students are treated as unique, different individuals in itself makes it a more Christian approach.
3. A math-lab approach forces the student to think for himself, initiate his own activities, make his own decisions This results in more relevancy and more independence from the teacher.
4. A math-lab approach provides for integration with other areas of the curriculum, and shows the students the unity of creation

reviewed from time to time in a variety of settings such as applications and games, and students are required to memorize only after they see the need for memorizing the concepts

On the other hand, we must ask ourselves certain questions about the math-lab approach even while putting it into effect: What kind of discovery, with what degree of teacher guidance, with what kind of learner, under what conditions? At what age level does knowledge become better directly transmitted by an expository approach? Is there any one specific tool or technique that provides the proper learning situation, or is a variety of approaches needed in any one classroom? What are the classroom and other physical limitations in implementing a lab approach? Is the teacher willing to introduce the approach, or is he likely to be ill-prepared and fall back into the "easy" textbook approach? How can we prevent a lack of proper continuity between elementary and secondary schools? Some of these questions we will not be able to answer until we have actually tried the math-lab approach for a period of several years.

In the implementation of a discovery program, there will be a period of adjustment, especially when a lot of traditional teaching has taken place. This suggests a gradual beginning for most situations. A mathematics corner could be begun, with the enrichment or remedial aspect in mind. This should lead to a sense of responsibility in the children towards their program.

There could also be total involvement, in which the active learning approach is used. The teacher would give no formal direction usually, but poses broad questions that lead to student investigation. This method is probably most effective when used once or twice a week.

For ideas as to how to implement the approach see Freedom to Learn (Biggs and Maclean); for concrete examples of the approach, see Mathematics in the Primary School (Biggs). Some general consider-

1. Emphasis needs to be laid on real discovery by the children. The experiences which are planned will depend on the ability and response of each individual child. A range of experiences will be required even for a small group of children.

2. Discussion between teacher and children is an essential part of the learning process: occasionally with the class, frequently with a group, occasionally with an individual child.

3. The experiences devised for the children need to be progressive. And the child must be aware that he is making progress.

4. The child's recording of his work, whether oral, written, expressed by a diagram or by a graph, must be his own. Considerable flexibility must therefore be allowed. Flexibility of method must similarly be allowed in problems. The first attempt must always be the child's own.

5. Computational processes. Even in this field first-hand experiences can lead the child to devise his own methods of written computation, in the first place. These can be refined later.

6. Records of the child's progress. Such records can only be the result of the teacher's close study of each child's number knowledge and the method by which he does even simple written computation. Only in this manner real discrepancies and basic weaknesses are revealed.

7. Books. In mathematics, as in English, a reference library of attractive books can be a very valuable asset and source of ideas for children and teachers.

8. The teachers. For the sake of continuity it is essential that every teacher should have a copy of the entire scheme of work. It is also important that teachers should have considerable latitude within the framework of the scheme.

9. Each one of the concepts included should be introduced at recurring intervals, at least once a year and many of them far more often, but each time in a new guise, with the idea of progression in mind.

10. It must be admitted that learning by discovery is a slower, although surer, process than apparent learning by instruction, followed by mechanical practice. By the age of nine, however, those that learned by discovery will have caught up with the written computational processes of the traditional procedure, so that their performance in this respect would be equal, if not surpass, that of children taught traditionally.

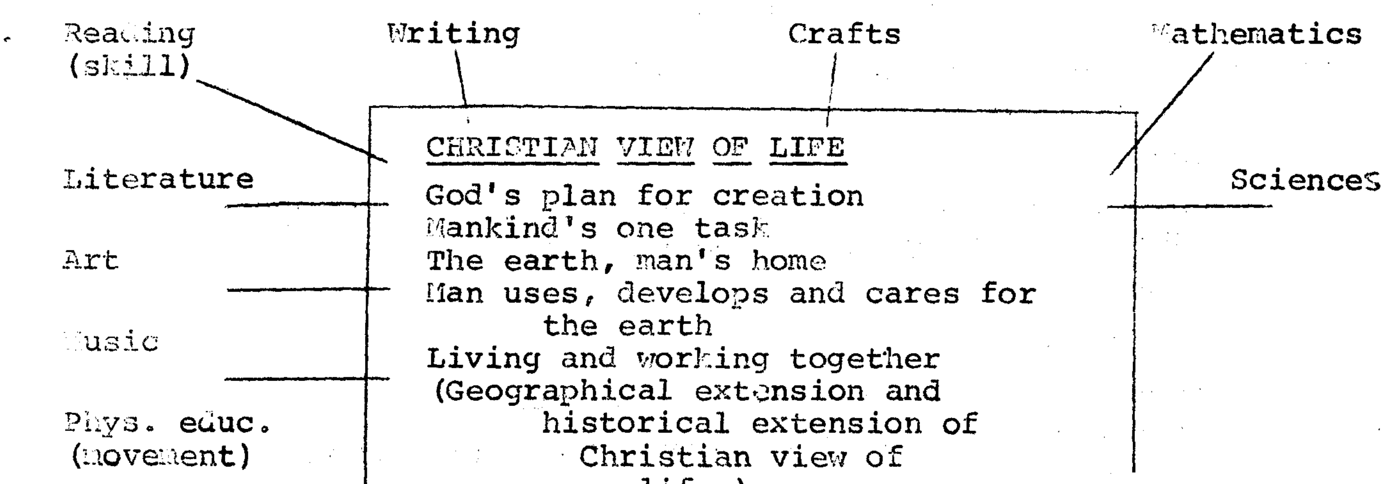
11. There is one aspect in which teachers would have no doubt about the superiority of the discovery method: the child's ability to think for himself and his confidence in problem solving, real or imaginary. When a child is accustomed to using real materials to solve problems, when he is used to making his own discoveries, he becomes self-reliant in this respect and often shows considerable initiative and invariably makes an attempt at any problem with which he is faced.

12. The discovery approach, in which the child is asked to explore a situation in his own way, is invaluable in developing creative and independent thinking in the child. His innate interests force him to concentrate on the creative problem at hand. It is clear, however, that at a later stage, the discovery approach is not expected to predominate.

13. It cannot be over-emphasized that, provided the children's first attempts at written computation are made as records of their own experience, when an efficient method is eventually reached, far less practice will be required to attain and maintain efficiency in this process. Many teachers have found that they can safely reduce this practice to two periods (and some to one) each week and that the children's efficiency in computation has improved and not declined in consequence.

14. The experience of many teachers has demonstrated that all the concepts listed in the summary can be discovered by the majority of young children themselves. It is not until about age 7 that the majority of young children are ready to systematize the number relationships they have discovered.

15. Since mathematics deals with the (abstracted) numerical, spatial and kinetic aspects of concrete things, the mathematical concepts learned by discovery from concrete experience can be fully integrated with the rest of the curriculum, particularly science, physical geography, crafts and art, and the learning of writing and reading.



The following is an outline of a useful book on the laboratory approach.

THE LABORATORY APPROACH TO MATHEMATICS. Kidd, Myers, Cilley. Science Research Associates, Inc. (SRA), 259 Erie Street, Chicago, Illinois 60611, U.S.A. (1970.)

"Before adopting a laboratory approach, teachers should have a fairly clear concept of what the method involves and should have given careful thought to and have tentative answers for questions such as the following:" (Foreword, p. ix)

- what activities should be used in class?
- what kind of curriculum materials should be available for student use?
- what is the role of evaluation and what should be the nature of reports to parents?
- what type of facilities should be provided for a mathematics laboratory?
- how can the approach be used so as to allow for individual differences among students?

The laboratory approach and special classroom (one that has been reorganized and equipped to allow for individualized learning) have the following characteristics: (pp. 11ff.)

- relates learning to past experiences and provides new experiences when needed
- provides a non-threatening atmosphere conducive to learning
- allows the student to take responsibility for his own learning and to progress at his own rate

Activities should be designed to provide the following: (pp. 2[illegible]ff)

- readiness
- concept development
- concept synthesis (should allow the student to review, organize and integrate mathematical ideas)
- recall
- application
- planning, evaluation, and remediation

Instructing students in the laboratory method: (pp. 33ff)

- identifying and stating the problem

- carrying through the plan of attack
- follow-up: drawing conclusions, comparing results, or comparing methods.

Ways of conducting a laboratory lesson. (pp. 42ff)

- teacher demonstration
- small groups on the same experiment
- small groups on different experiments

Selecting laboratory investigations. (pp.61ff)

- relation to the goals of instruction
- proper level of difficulty
- interest to students
- relation to available tools

Space and facilities for the laboratory. (pp. 93ff)

- more space per pupil than conventional classrooms
- flexibility in arrangement and use of classroom equipment
- soundproofing of classrooms
- work bays and study carrels
- technical equipment
- adequate lighting and electrical outlets

Evaluation. (pp. 129ff)

- evaluation of: student, teacher, materials, and approach
- tests: should be diagnostic (to find students' weaknesses)
- observation of a student involved in laboratory activities can reveal his psychomotor development
- should include student-teacher conferences

Quotations:

"Most teachers who are successful with the laboratory approach begin gradually. They usually make only partial use of this approach at first. The whole class does the same investigation or someone gives a demonstration to the whole class." (p. 21)

"Any planning for laboratory facilities must take into account the goals of instruction and the limitations under which the teachers operate." (p. 112)

Conclusion:

- a chapter on mathematics for the low achiever is also included

(c) Some notes on problem solving

Problem material should be considered one of the essential features of the curriculum: topics can be introduced through the posing of a concrete problem, and students learn mathematics through the solving of problems. The composition of meaningful problems is one of the largest and most urgent tasks in curricular development. In fact, we would hope to convey to the student that each mathematical idea appeared first as the solution of some problem by some person. Good problems are those that do some of the following: point out a significant historical development, relate to a variety of other problems, can be used in a variety of other problems, can be used in a variety of other areas, do not get bogged down in a great deal of "excess" (although there should be some "excess" on occasion), have more than one way of attack, generate computational skill, show something about the structure of reality. The problem must be at the child's level and he should normally have enough information to solve the problem (but not always). The child must be interested (i.e., motivated) so that he accepts the problem as his problem. Students should sometimes be shown problems for which no solution exists (e.g., Fermat's, Goldbach's, Hamilton's, or the 4-colour problem).

Problems can be given at various levels:

1. Recognition and recall (knowledge)
2. algorithmic thinking, generalization (Comprehension and application)
3. Open search (analysis and synthesis)

There is a danger in emphasizing low-level objectives such as simple recall and comprehension and neglect an important aim of learning mathematics - the ability to apply its methods to new situations. While not all students will be able to reach the levels of analysis and synthesis, the opportunity must be there for students who are able to do so.

To induce higher-level problem solving, Avital suggests that (1) good comprehension is essential, (2) that we use multiple models

(3) we expose students to problems at all levels in teaching and testing, (4) we emphasize generalizable strategies, (5) we teach procedures, not formulas, and (6) we teach through problems, leading the students to ask appropriate questions.

In conclusion, we list some implications of research in mathematics education for problem solving in the classroom.

1. In problem solving, it has been shown that (i) students react best to a <u>variety</u> of problem settings, (ii) problems <u>relevant</u> to the interests of children are most effective, (iii) students should be taught to make use of mathematical sentences whenever possible, and if they have difficulty, of auxiliary diagrams and drawings.
2. From the various studies in <u>Einstellung</u> (rigidity), the curriculum must reinforce or at least allow attempts to vary strategy in seeking solutions and when the text provides a series of problems, it should give an occasional problem which demands the application of a different principle.
3. We must keep in mind that the reading ability of students is positively correlated with their ability to solve problems. Remedial reading instruction has been shown to have a positive effect on arithmetical computation achievement. This may be caused by a number of factors, but we must be concerned with the readibility of both text material and problem material, and design the curriculum so that students are helped to learn to read mathematics.
4. Children differ a great deal in the extent to which they rely on direct and auxiliary processes. Some good problem solvers usually rely chiefly on direct translation when solving a "word" problem: others make extensive use of auxiliary ones and physical representation.
5. Some practice ("drill") is necessary to acquire speed and accuracy in the manipulation and the use of algorithms. Practice is most effective when (i) the student wants to improve (ii) it is done in short periods spaced out over a period of time, and (iii) the student is kept aware of his progress

CHAPTER III

THE MATHEMATICS CURRICULUM

1. Some Preliminary Considerations

In designing the curriculum, we must keep in mind the following:

a) The purpose of the school is that the child learns to discern truthfully in order to gain a deeper understanding of his many-sided religious tasks in life (and not for personality reasons, which are byproducts of good teaching).

b) Our understanding of creation determines our perspective. To gain true insight, we must be aware of the structural laws of God's creation.

c) Our theories must point back to reality. The study of mathematics ought to enable the child to be a better baker, businessman, etc. There's more to it than the child just learning the manipulation of formulas. Nor are we just preparing the students for college or other post-secondary institutions: We are preparing the students for life.

d) In the future we must develop an overall curricular design that includes all of the individual components and their relationships. The curriculum must be viewed as a whole rather than as bits and pieces. This must be developed simultaneously with work in individual subjects. The strategy of attack on a broad front must necessarily be complex, but development, appraisal, and reorganization of the curriculum as a totality, rather than as collection of pieces, are called for. A separate-piece approach creates the danger that those significant mankind problems growing out of where and when and how one lives - problems which cut across subject lines - may not be brought into the classroom, and as a consequence, the child is not truly prepared to face his many-sided tasks in life.

2. A Brief Outline of the Evolution of Recent Mathematics Curricula

The main factors leading to reform of the mathematics courses in the United States are:

a) Extensive mathematical and scientific illiteracy among high school graduates post-WW II.
b) The cold war called for knowledgeable persons in mathematics
c) The middle class saw education as the key to the good college and hence to the good life
d) Values were crumbling and shifting
e) Because of the knowledge explosion, the curriculum was seen to need fresh content and comprehensive reorganization
f) Researchers were experimenting more with children's ability and methods of learning

Two general feelings resulted:
a) That more advanced training should be provided for more students at an earlier time
b) That the complexity of the mathematical applications and the rapidity with which they were being developed and changed signified that advanced training should be carried out with attention to meaning and understanding.

The university of Illinois Committee on School Mathematics (UICSM) became the first major new project. Understanding was the main goal of the program, which was accomplished (supposedly) by the use of much rigour, and much precise language. One advantage of the program was that it presented a unified approach, and did not separate the branches of math as much as most U.S. courses.

The College Entrance Examination Board influenced the high school curricula indirectly but significantly by setting standardized examinations based on: (i) concepts and skills for college math preparation, (ii) deductive reasoning both in geometry and algebra, (iii) appreciation of structure, (iv) sets, variables, functions, The School Mathematics Study Group has had more direct influence on the curriculum in the U.S. than any other program, although its philosophical direction differs from one group of texts to the next.

Ralph Tyler of the University of Chicago has been influential in the basic approach used in developing many modern U.S. curricula. He raises four basic questions:

1. What educational purposes should the school seek to attain?
2. What educational experiences can be provided that are likely to attain these purposes?
3. How can these educational experiences be effectively organized?
4. How can we determine whether these purposes are being attained?

His subordination of a philosophy of knowledge and of education and a theory of learning to "screens for selection and elimination among possible objectives" has been partly the cause for the present lack of cohesiveness in purpose in many modern mathematics texts: philosophical views are incorporated subconsciously and psychological considerations are given superficial lip-service at best. In fact, it is assumed in most math curricula that the ends and means of schooling are derived first from the academic discipline and only secondarily from characteristics of children or youth and of society in general. One reason that new math programs have not been as successful as had originally been hoped is that these factors have not been considered, and that at the same time the goals and objectives of the programs have not been made explicit (except for the content to be covered). Such objectives, of course, must be consistent with the philosophy guiding the curriculum - philosophy is the common denominator of all programs and activities.

Present trends include the following:

1) more curricula for all learners, not just the college-preparatory stream. More concrete materials are being introduced at the elementary levels and in the programs for slow learners. However, almost all books for average and above-average high school students are still highly abstract, with few applications that show the rightful place of mathematics.

2) More attention is starting to be paid to methodology and theories of learning; again, the elementary curricula are ahead of the high school curricula in this respect (e.g., compare the emphasis on methodology in the Arithmetic Teacher with the stress on content in the Mathematics Teacher.)

3) There is a trend - though it is gradual - toward more unified courses.

Developments in Great Britain have been more encouaging. At the elementary level, the simultaneous development of the Nuffield Project and the work of Edith Biggs (Mathematics in the Primary School) have led to the widespread introduction of a math-lab approach using concrete materials. At the same time, these programs have considered the nature of learning and the characteristics of the child far more thoroughly than the content-centred U.S. programs. There is much that is worthwhile in these programs, although the philosophical and psychological basis of the Nuffield program is to be questioned (e.g., their emphasis on sets and their whole-hearted acceptance of Piaget).

For secondary level, the School Mathematics Project is the most significant series. Its authors have laid down the following basis for the program:

1) The content of the school course should be modified by the introduction of fresh material, some of this being comparatively new in mathematical history, and some being much older but now having an importance which it lacked before. Some traditional material needs rethinking, and teaching in a new way...the teaching of mathematics needs to be better integrated with contemporary applications in industry and research, and, at the same time, be able to evoke a mathematical response from the pupil.

2) In the teaching of mathematics, there must be a proper understanding of the relevance of recent psychological investigations, and teachers must understand this new knowledge and use it as the basis for a technology of teaching.

3) Mathematics does not start with the finished theorem; it starts from situations. Before the first results are achieved there must be a period of discovery, creation, error, discarding and accepting.

4) The important ideas of advanced mathematics such as vector, or closure in a group should be introduced early in the curriculum in a primitive form.

5) We repudiate any suggestion that "modern mathematics" is to be equated with the axiomatic method. In teaching, a set of axioms can only be understood after a long, initial, quasi-experimental investigation. Traditional Euclidean geometry should not be included, since (i) Euclid's axioms are deficient, and (ii) a proper axiomatic approach is extremely awkward.

The courses are unified, although the spiral approach used is somewhat disjointed. The basic unity of mathematics is shown by interrelating a wide variety of topics. There are many concrete situations in these texts which are worthwhile incorporating in the mathematics curriculum, and the approach impresses on the student that mathematics is something that deals with reality.

In Canada, most texts have followed the approach used by the U.S. curricula, but incorporating some of the British features. Unfortunately, this copycat approach has led to a disunified program in most cases. There are a few series that are no worse than the average U.S. text (e.g., Contemporary Mathematics, and the Gage series).

The significant work in Canada in mathematics curricula has been done mainly by "imports". Edith Biggs spent a year in Ontario showing how her approach could be implemented in the elementary schools. The work of Dr. Dienes at the University of Sherbrooke has found widespread acclaim throughout the world, but little in Canada. His books and programs give many ideas for a math-lab approach - from a Gestaltist point of view.

3. Some Notes on Geometry

Dieudonne made the oft-quoted statement: "Euclid must go!" But he made more positive remarks, and they are worth repeating:

1. "Nobody need be concerned, in secondary schools at least, with teaching the future professional mathematicians (not to speak of the great ones), of which there may be one in 10,000 children. What is really at stake is the kind of mental picture of mathematics that will emerge in the mind of an average intelligent student after he has been subjected to that treatment for several years."

2. "A mathematical theory can only be developed axiomatically in a fruitful way when the student has already acquired some familiarity with the corresponding material - a familiarity gained by working long enough with it on a kind of experimental, or semi-experimental basis, i.e., with constant appeal to intuition."

3. "When logical inference is introduced in some mathematical question, it should always be presented with absolute honesty - that is, without trying to hide gaps or flaws in the argument."

4. "Geometry should put the emphasis not on some artificial playthings as triangles, but on the basic notions such as symmetries, translations, composition of transformations, etc."

5. "Whenever possible, any notion should be developed both from the algebraic and the geometric point of view. Throughout geometry at high school level, the emphasis should be on the linear transformation, their various types and the groups they form."

6. "The curriculum at high school level should deal only with mathematical objects that have an immediate intuitive 'interpretation' of some kind."

He adds that "My quarrel is with the methods of teaching geometry, and my chief claim is that it would be much better to base that teaching not on artificial notions and results which have no significance in most applications, but on the basic notions which will command and illuminate every question in which geometry intervenes."

"For instance, whereas the notion of vector has paramount importance everywhere in modern science, the notion of triangle is an artificial one, with practically no applications outside the highly specialized fields of astronomy and geodesy."

The Royaumont Seminar at which this address was given led to the formation of a Group of Experts brought together by the O.E.E.C., who prepared an outline of a syllabus for modern treatment of the mathematics curriculum. For geometry, they stressed the following principles:

1. No hard and fast terminology should be employed in the first stages. New words should be defined within the context in which they are used.

2. A physical model (giving rise to observation and experience) is the basis from which mathematical abstraction is developed.

3. It is essential that the pupil learn to think creatively and intuitively. He must be given opportunities to find his own problems, to state his own solutions.

The topics suggested by the group for lower secondary levels included:

1. vectors as directed line segments
2. angle-properties of angles in connection with parallel lines, polygons, circles;
3. transformations studied from a physical, intuitive standpoint to investigate the properties of figures;
4. simple graphical transformations;
5. non-metric properties of the line and the plane; and
6. the use of short "logical chains" to justify some of the properties of figures previously investigated from an intuitive basis.

For the last three years of high school, the topics for study that they suggest include:

1. groups of transformations
2. affine geometry (including vectors and vector spaces)
3. conics (including projective and descriptive geometry); and
4. finally, axiomatic treatment of one or two topics such as vectorial space or Euclidean metric space or synthetic Euclidean geometry.

This type of curriculum, if integrated and well-structured, would be much superior to any of the present texts written in Canada. These suggestions of the group as well as the ones by Dieudonne also tie in with what the O.I.S.E., K-13 Geometry COmmittee wrote in its report (O.I.S.E., 1967):

> "Visual and intuitive work are indispensable at every level of mathematics and science, both as an aid to clarification of particular problems, and as a source of inspiration...it would be inappropriate at any stage to emphasize axioms too heavily or to overstate their role...geometry may not even be the best branch of mathematics for the illustration of the use of axioms."

With respect to teaching geometry as an axiomatic, deductive system, the same authors write:

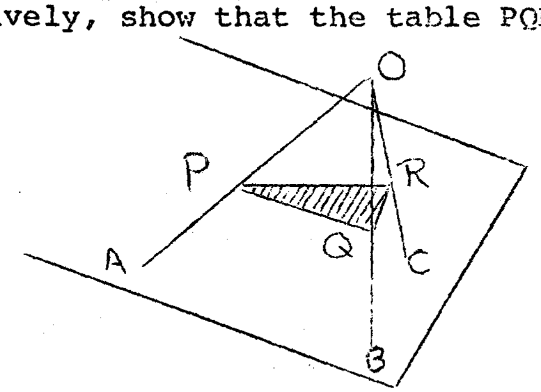
> "...in some ways geometry is the least suitable branch of mathematics for conveying the ideas of proof. A few experiments may be sufficient to convince a student that a result in geometry is true, so that his motive for seeking a proof is weakened. Second, even when he searches for a proof he may have great difficulty in distinguishing between relations which he can see are true, and those which he has succeeded in proving deductively from given assumptions. Third, Euclid's geometry is not, in fact, a deductive axiomatic system: Euclid gives definitions of point and line which he uses nowhere in his later work, and on the other hand makes use of properties which he has not stated...it seems we should teach geometry for its results, and as an exercise in <u>informal reasoning</u>. It is perhaps worth pointing out that the discoverers of non-Euclidean geometry...did their work <u>before</u> the logical gaps in Euclid had been plugged."
>
> (O.I.S.E., 1967, pp. 3 - 26)

After pointing out the advantages of teaching motion (or transformation) in geometry, the report concludes that "it would be wise to keep in the curriculum some <u>informal</u> geometry, borrowing a part of Euclid's ideas together with the other, more recent methods of attack". One of the reasons given is that many results concerned with angles, having fairly simple proofs along Euclid's lines, can be very difficult to prove by co-ordinate, vector/matrix, or transformation geometry. Also, if geometry is dealt with informally, by plausible reasoning rather than by strict proof, it is possible to deal with the subject in a much swifter tempo, which enables the student to grasp much sooner the basic unifying concepts of geometry and how the numerical aspect of geometrical situations can deepen insight into problems, and leads the student much more quickly to significant applications of geometry in everyday situations.

In short, geometry must be developed from an experimental, intuitive approach, showing clearly how numerical and algebraic techniques can be used in its development (e.g., using matrices to describe transformations, analytic geometry). We should not neglect Euclid's method where this approach is advantageous. A variety of mathematical techniques and a variety of applications can be used to point out the unified structure of mathematics.

That a different type of approach to geometry from our present one is possible is pointed out in the following examples, which might come in a section on the use of vectors in solving everyday problems: (adapted from the SMP texts)

1. A tripod with unequal legs OA, OB, and OC is standing on a horizontal floor (see figure). The legs are held rigid by a small triangular table PQR attached to the midpoint of each leg. By taking OA, OB, and OC to be 2a, 2b, and 2c, respectively, show that the table PQR is horizontal.



2. In coming from a coal face of a mine, coal follows the route ABCDEF to the railway trucks. The route is described by vectors defined like those in question 3.

$$\vec{AB} = \begin{pmatrix} 0 \\ -180 \\ -8 \end{pmatrix}, \quad \vec{BC} = \begin{pmatrix} 140 \\ -140 \\ 2 \end{pmatrix}, \quad \vec{CD} = \begin{pmatrix} 0 \\ 0 \\ 220 \end{pmatrix},$$

$$\vec{DE} = \begin{pmatrix} 0 \\ 36 \\ 6 \end{pmatrix}, \quad \vec{EF} = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}.$$

(a) If D is at ground level, how far below ground level are the men working at the coal face?
(b) Describe in words the path of the coal underground.
(c) What is the position of F in relation to A?

This type of question could lead to several investigations. In the first, one could ask, under what conditions will a chair be stable on a floor? For the second question, one could investigate whether other routes would be possible, what other mathematics problems one might come across in a mine, etc.

4. What Content Should Be Included in the Curriculum?

The seminar did not discuss this question in detail, since not enough research has been done in finding suitable topics which will enable us to meet the objectives we have set forth for the mathematics curriculum. At the elementary level this is somewhat easier to determine than at the secondary, since it is in the early stages that the "mathematics for everyday life" must be taught. Below is a summary of topics that Dr. DeGraaff prepared, and that was felt to be suitable by the seminar for the elementary level. It should be stressed that the topics should be interrelated throughout the curriculum.

Summary of the mathematical concepts most children can learn by the age of seven (Piaget's pre-operational stage, 2-7 yrs; intuitive thought, 4-7 yrs.).

1. Comparing quantities of objects: learning the language, and later the symbols, of inequality: greater than, less than, not as many as, too many, too few, not enough; and equality (matching or one-to-one correspondence): the same as, is equal to.

2. Counting quantities of objects: (Cardinal number): conservation of number; composition of numbers up to 20 known without counting on or counting back.

3. The number line (ordinal number): numbers in order up to 100; acquaintance with numbers beyond 20, but no written manipulations of these numbers in isolation from experience; growing awareness of place-value in number notations.

4. Measurement: conservation of measures; knowledge of common units of weights and measures which normally come within the experience of young children; counting of money.

5. Simple fractions; learning the language, and later the symbols of simple fractions: one half, one quarter, three quarters.

6. Addition, Subtraction, Multiplication, and Division; varied aspects of these operations as they arise in real classroom situations.

7. Shape and Size; the properties of shapes: dimensions, symmetry, similarity, and mathematical limits, which young children can discover for themselves.

8. Pictorial and graphical representation; of quantity relationships and spatial dimensions; pictorial record of counting and measuring; block graphs.

Summary of mathematical concepts most children can discover and master during the intermediate level (Piaget's operational stage 8 - 12 yrs.).

Number relationships:

1. Comparing quantities of objects; matching.

2. Counting; cardinal numbers; conservation of number; written numerals.

3. Numbers in order; the number line; ordinal numbers; combining numbers up to 20.

4. Place-value; other ways of representing numbers (Roman numerals, Egyptian numerals), etc.

5. The operations of addition, subtraction, multiplication and division based on varied concrete experience, recording at first in the child's own words; symbols then used as shorthand; finally practice.

6. Extension of number knowledge to include numbers between 20 and 100; the four operations: addition, subtraction, multiplication and division applied to numbers; secure knowledge of number trios; multiplication and division tables.

7. Place value number bases other than 10; remainder arithmetic.

Children's own problems; computational practice and story arithmetic.

Extension of the idea of numbers; fractions and decimal fractions based on concrete experience; the four operations using these numbers based on first-hand experience; directed and negative numbers.

Measurement:

1. Concrete experience of all kinds of measurement and use of money, length, weight, capacity, time, temperature, speed;

2. First-hand experience in the four operations with standard units of measurement.

The Priorities of Geometric Shapes:

1. Geometrical shapes and the relationships between the sides, angles and diagonals; solid shapes: cubes, rectangular boxes, cylinders, cones, spheres; making the regular solids; regular flat shapes: square, rectangle, triangle, hexagon, octagon, pentagon; shape fitting; tiles; circle.

2. Angles; square corners or right angles of squares and rectangles; angles by rotation; compass directions; intersecting lines and vertically opposite angles; parallel lines and intersecting lines and angles so formed.

3. Conservation of area, volume; perimeter, area, volume of irregular shapes; the relationship between the perimeter and side of a square; circumference and diameter of a circle; the area and side of a square; the area and diameter of a circle; the volume and edge of a cube; the volume and diameter of a ball.

4. Symmetry; recognition in 3 dimensions and 2 dimensions; recognition of the types of symmetry: reflection, rotation, translation.

5. Similarity; maps, scale models; recognition of the precise relationships: angles equal, sides of the same ration; how squares grow; how cubes grow.

6. Mathematical limits from shapes: spirals, contracting and expanding squares.

Graphical representation:

1. Collecting data and recording these by real objects or by symbols, or by pictures.

2. Collecting data and recording by block and column graphs.

3. Averages: ratio, proportion and rate; simple statistics, percentages.

4. Relationship between two variables; data recorded by graphs.

5. Types: constant ratio graph, straight line; constant product (hyperbola); squares (parabola); cubes; growth curves.

6. Positive numbers and negative numbers; directed numbers.

For the high school curriculum, it was suggested that the following topics ought to be included (this is a very preliminary list not based on a thorough study; there may be other important topics that we overlooked):

Basic concepts to which all of the listed topics should relate:

Discreteness (number)	Continuity (space)

1. [illegible] ss vs. continuity)
2. Mathematical relations and functions
3. Linear algebra (transformations, vectors, matrices)
4. Probabili[illegible]y and statistics
5. Calculus
6. Topology and transformations
7. Analytic geometry
8. Trigonometry
9. Approximation theory (incorporated elsewhere?)

This list is not meant to list all possible topics, and we envision a unified curriculum in which the above topics are interrelated. The curriculum should incorporate some set notion (but the notion of set must not be presented as a basic one in mathematics), and flow diagrams may be introduced at appropriate places.

5. Examples of the Way in which the History of Mathematics Can Be Integrated with the Curriculum

We do not wish to teach a course in the history of mathematics *per se* during the student's mathematics courses; rather, our aim is to show through historical examples that mathematics is a developing science; was influenced by the cultural forces at work in various civilizations, and in turn influenced the development of civilizations. We give outlines of two examples that could be developed in a mathematics curriculum.

a) The Development of the Concept of Real Numbers

The cultural ideal of the Greeks centred about true excellence, taking possession of the "beautiful", harmony, balance, a respect for the wholeness and oneness of life. They viewed the cosmos as a whole, and had a sense of the organic structure of life. The Greeks were always looking for one law pervading everything and tried to make their life and thought harmonize with it.

The Pythagoreans thought that the basic concept which unified and could structure their world was that of natural number. Everything was explained in terms of this: there were numbers representing man, woman, the family, and interrelations were explained in terms of the numbers assigned to the concepts. Rational numbers fit into this pattern: they were just ratios of natural numbers (e.g., if the length of strings of a musical instrument are in the ratio of 2:3, a pleasing chord results).

However, the Pythagoreans discovered that the diagonal of a square of one unit in length is not equal to a rational number, and they proved that numbers such as $\sqrt{2}$ cannot be written in the form a/b. Thus they showed that you cannot subdivide a length of one unit on the number line into small units that also subdivide a length of $\sqrt{2}$, i.e., there is no small unit that divides evenly into both lengths. Thus we call the numbers incommensurable.

This was a devastating discovery for the Pythagoreans. It was as if their world of order and harmony had been destroyed. Natural numbers could no longer explain everything in the world; their reductionism had failed. How devastating this was is pointed out by the legend that tells about the mathematician who knew this result and who returned to Greece from Italy (where the Pythagoreans worked and studied). It was arranged that he would be thrown overboard and drowned on the way so that he would be unable to let this out of the bag in Greece!

From then on, Greek mathematicians did not develop the numerical aspect of mathematics to any extent. The emphasis was on geometry, partly because of the awkward notation the Greeks used for numbers, but also because they did not solve the problem of incommensurability and therefore tried to explain the universe in terms of geometrical concepts, after the time of Pythagoras.

This historical development could be incorporated in the unit dealing with real numbers, and can be used to point out the irreducibility of the concepts of discrete number and that of continuity, and that the one cannot be explained in terms of the other.

b) <u>Numeration systems</u>
This topic would be introduced at the junior high level or upper elementary level. The goal of the approach is an analysis of our own number system and an understanding of other possibilities of numeration.

<u>Suggested Procedure</u>

1. Ask the students to suggest a new written code for expressing numbers.
   - compare and contrast the suggestions:
     a) what type of operation involved?
        additive(?) IIII
        subtractive IX
        multiplicative Chinese-Japanese

     b) how many symbols used?
        units, tens (or any other base), 100's
        0 - 9; 10,20, --- 90; 100,200 --- 900
        0 - 9; and combination through positions
        a new symbol for every number?

     c) what groups these symbols?

2. Draw analogy between their systems and a similar historical system--this could include:
   Egyptian hieroglyphics
   Babylonian cuneiform

Greek (Attic & Ionic)
Roman
Chinese-Japanese
Mayan
Hindu-Arabic

a readable summary is given in *An Introduction to the History of Mathematics* by Howard Eves

3. Perhaps the class could divide into groups--each working with a particular system
   - practice writing numbers
   - some simple computation

4. Discuss the merits of each system including the Hindu-Arabic. What are some good criteria for a number system?

Comments

1. Try to show that the present system was one of many attempts at numeration, (such as they were originally asked to do) (a symbolism for discreteness?)

2. Discuss what led to general acceptance of the present system.

3. This effort only implements historical ideas out of context of the cultural situation--but yet it helps to introduce and explain our number system.

6. Some Suggested Activities for a Math-Lab Approach in the Elementary School

The following pages contain some activities and suggestions for implementing a math-lab approach in the classroom. This list is just a beginning; for more examples consult the books in the bibliography or use your own ideas.

MATERIALS

- Counters: buttons, beans, peas, bread tags, beads, tops, chestnuts, marbles, etc.(USE YOUR IMAGINATION!!!)
- balances (suggest buying one or two accurate ones, but have several home-made ones as well. Include one spring balance.)
- weights: a large variety, varying in size
- commodities for store (boxes marked with prices, etc.)
- number lines
- abacus
- ribbon, string, straws: for measuring
- yardsticks, rulers (Some marked in inches, feet, yards,
  some marked in metric system
  some unmarked)
- tape measures

- rope
- empty fruit tins of various sizes
- plastic measuring cups
- funnels and plastic tubing
- rectangular cylindrical & other containers of various shapes
- set of shapes (square, cube, triangle, rectangle, etc.)
- clocks (clear figures and movable hands)
- egg timers
- home-made pendulums
- stop watch
- calendar with large figures
- thermometers
- play money and purses with real money
- blocks
- trundle wheel
- buckets; plastic ones make less noise!

This list is by no means complete. But it does give an idea of materials which could be used.

GAMES

1. ROLL-A-NUMBER

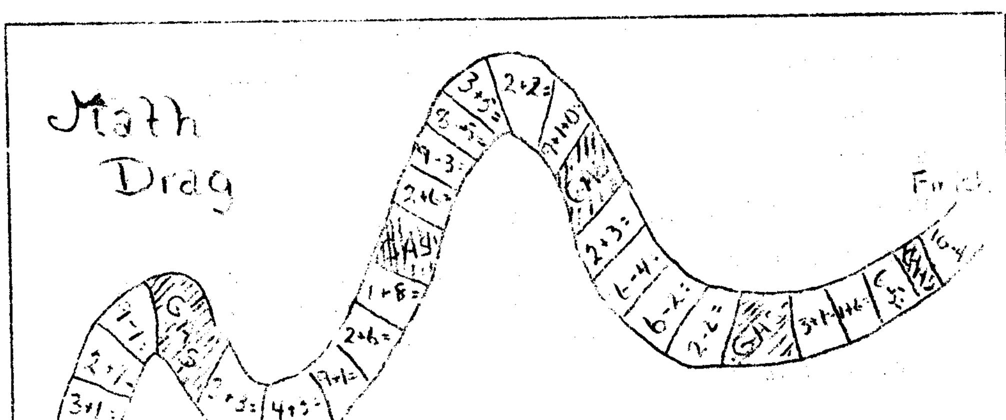
Two dice, one numbered 1, 1, 2,2,3,3,; the other 2,3,4,5,6,7

Three each of cards numbered 1,2,3,.......10

The children are each given 5 cards. Each child in turn rolls the dice and plays cards from his hand to match the numerals on the dice. The first child void of cards is the winner. The game is repeated.

*

2. MATH DRAG



One die

One marker per child

Ten gas cards, made of 2" x 3" bristol board and marked with one of +1, -1, +2, -2, +3, -3, +4, -4, +5, -5.

Game board (see diagram above)

Each player rolls the die and moves forward the number of spaces indicated on the die. He must correctly complete the card on the space to stay there; if he cannot, he goes back 3 spaces. If a player lands on a GAS space, he draws a card and moves the number of spaces indicated on it.

* *

3. SNAKES AND LADDERS

dice

one marker per child

game board (structured as commercial game, with number stories in each space)

Each child moves according to number on die. If he answers the sentence correctly, he may remain there. If he answers incorrectly, he must move back to where he was before. If he lands on a ladder, he must correctly answer the number sentence on top and at the start of the ladder before he may climb the ladder.

4. <u>FACTO</u>

F	A	C	T	O
4-4 =	3+4 =	2+2 =	6+2 =	9-6 =
9-4 =	4-2 =	4+5 =	10-3 =	7-1 =
3+3 =	5+3 =	Free	2+0 =	10-6 =
2+1 =	10-1 =	5+0 =	10+0 =	3-2 =
1+1 =	6-3 =	9+1 =	7+0 =	4-0 =

Players' board for each player (each board should contain a different arrangement of expressions.)

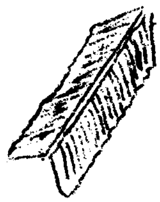
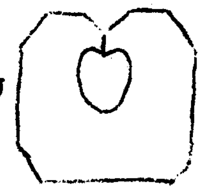
Eleven pairs of cards numbered 0,1,2,3,4,5,.....10

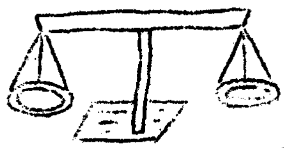
Quiet counters

Facto is played like Bingo. One child acts as the caller; the others each have a player's board. The <u>Free</u> space is covered with a counter. The caller chooses a number card, calls out the number, and places it on the correct space on the board.

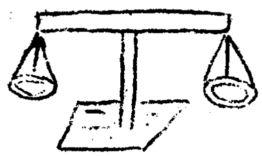
The first child to cover 5 spaces in any straight line is the winner.

<u>Some Samples of Activity Cards for Grade One</u>

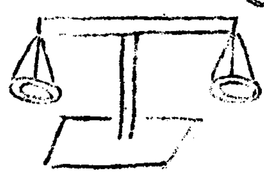
How many feathers  does a tag  weigh?

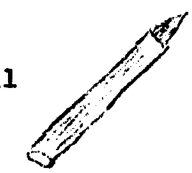


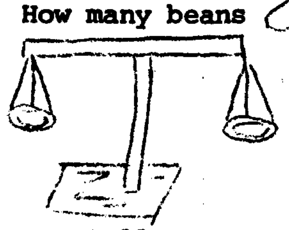
How many beans  does a car  weigh?



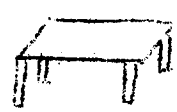
How many peas  does a stone  weigh?

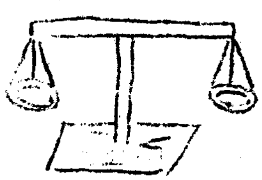


How many beans  does a pencil  weigh?



How tall are you? Use the number line on the wall to find out.

How many blocks  long is the table? 



How many beans  does a block  weigh?

CODE

A	-	ADDING
A	-	AREA
Av	-	AVERAGE
C COINS	-	MONEY
D	-	DIVIDING
G	-	GRAPH
M	-	MEASURING
M	-	MISCELLANEOUS
M	-	MULTIPLICATION
S_H	-	SHAPES
S	-	SUBTRACTING
T	-	TIME
V	-	VOLUME
W	-	WEIGHT

M - MISCELLANEOUS

C - MONEY

Take pictures or items out of catalogues or other written material. Paste these one cards and make problems about them.

Kellogg's Corn Flakes 16 oz. Pkg. 37¢

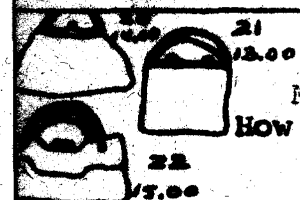
Dino bought a package of Corn Flakes with a $1.00 bill. How much change did he get?

Dad bought a crystal chandelier for $49.95. He paid with a $100.00 bill. How much change did he get?

Mother bought two pairs of skates. One pair was $29.95; the other $39.95. How much did she spend?

An Ad. from Cadet Cleaners

Mother wanted her dress cleaned. She gave the lady 75¢. How much change did she get?



Mother bought purse number 20. She paid with a $20.00 bill. How much change did she get?

---

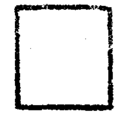
guitars from D to H. from .49 to 46.95

Paul bought two guitars, the H and F. How much did he pay for both of them? Put your answer in your math book.

---

pairs of shoes, prices ranging from 5.99 to 6.99

Mrs. Brown bought six pairs of shoes for her daughters. How much did she pay for them?

Take four toothpicks and make this shape with them. 

Then make a shape like this: 

1. Did any of the toothpicks become shorter? Why? or Why not?

---

1. Write in your notebook your guess of the length of six different things in this room.

   Now measure the length of each one.

   2. Write down your answers beside the guesses.

---

1. Measure the size of your foot.
   2. Show how they compare with the others in your group in an interesting way.
   3. Whose is largest?
   4. Whose is smallest?
   5. Does feet size have any relationship to the height of a person?

Measure your waist. Find other things in the room which are the same length.
Record.

Measure each of the following.
Estimate your answer first.

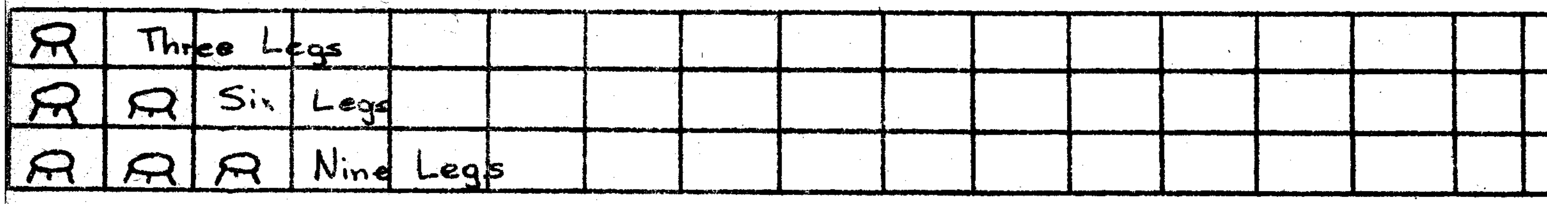
1. Width of floor tile.
2. Length of shoulder to fingertip.
3. Friend's height.
4. Length of table.
5. Other [illegible] objects.

Record.

G – GRAPH

Some farmers sit on three-legged stools and milk their cows by hand. Take a large sheet of paper and draw some stools, saying how many legs they have altogether.

1. Like this:



Keep going until you have drawn twelve three-legged stools.

It is interesting to notice that the farmer has two legs, the stool has three legs, and the cow has four legs.

2. Try to draw the same kind of picture for the farmer.

3. Now take a third sheet to make a picture for the number of legs of cows.

---

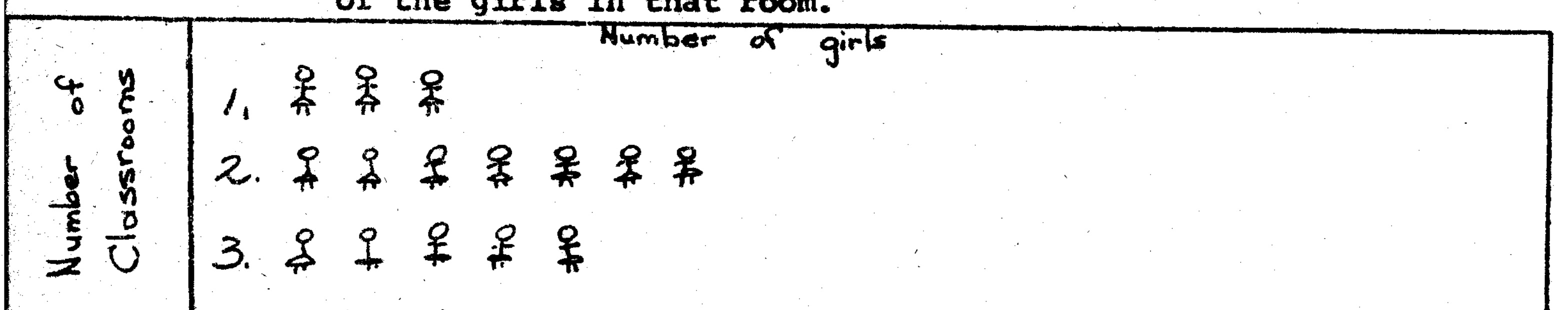
1. How many girls are in the classroom?
2. How many boys are in the classroom?
3. How many more boys do we have than girls?

Would you be able to do the same for some of the other classes in the school? Be sure to talk it over with the teacher before you decide what to do.

<u>For the other classes:</u>

4. Compare the number of boys to the number of girls. Do this for each class which you have chosen.
5. Look at the number of girls in each classroom. ~~In~~ classroom 2 put down a picture of a girl for each of the girls in the class. Make a row of these girls.

Then take your next classroom and make pictures for each of the girls in that room.



6. Which classroom has the most girls?
7. Which classroom has the least number of girls?

Can you do the same for the number of boys in each classroom? Try it.

Av - AVERAGE

Give one friend 3 elastics, another 5, another 6, and hhe third friend 10 elastics.

1. What must you do so that each person will have the same number of elastics?

Write in your notebook what you found out.

HOW MANY SQUARES?

On a small piece of cardboard make a square with all sides one inch.
Cut it out. This is one square inch.

1. On paper draw a shape six inches long and four inches wide. What is the name of this shape?
2. Draw in all the square inches.
3. How many times will your square inch fit into the shape you made?
4. Make other shapes that have the same number of square inches.

Take a tin can and a piece of coloured paper.
Find a way for putting the coloured paper around the tin.
The piece of paper should fit the size of the can.

1. In your notebook write down how you did it.
   Let a neighbour read how you did it. Then he has to follow your directions to do it on his can.
2. You take your neighbour's directions and follow them using a different can.

Take a sheet of graph paper and cut out some squares with sides of

one inch	four inches
two inches	five inches
three inches	six inches

Write on each square the number of little squares it has.

1. Now try to find out the distance all round each square you have cut out.

A - ADDING

MAGIC SQUARE:

Cut out some squares of cardboard, 1¼ inch square, enough to write the numbers from 8 to 16. Put one numeral on each card.
Now try to arrange these cards in the form of a magic square. You don't have to bother with the diagonal lines but if you can get these right as well, so much the better.

MAGIC SQUARE:

4	9	2
3		7
8	1	6

In a magic square, each vertical row of numerals, each horizontal row, and each diagonal row gives the same total.

1) Copy the whole square in your book.
2) Write the middle numeral in a different colour.

vertical - going up and down ↕

horizontal - going across ↔

diagonal - from a top corner to a bottom corner
(or from a bottom corner to a top corner) ↘ ↗

MAGIC SQUARES:

9	10	5
4	8	
	6	

The missing numbers in this magic square are 12, 11 and 7.
Write each missing number on a piece of paper the same size as the square.
First try to see if you can put the papers in the right places.

1. Copy the magic square in your notebook.
2. Write the missing numerals in a different colour.

MAGIC SQUARE:

16		2	
	10		
9	6	7	12
	15	14	

First try to find the numerals which have been left out of this magic square.
1. Now copy the whole square in your notebook.

D - DIVINING

Measure three feet of yarn. Cut it off at the 3 ft. mark.
Share this yarn with four people.
Each piece must be equal to the others.

1. Write down how you found out how to do it.

Take four toothpick boxes and some toothpicks.
Share the toothpicks among the four boxes.

1. How many toothpicks did each box get?
2. What is an easy way of finding out how many toothpicks you have altogether?

Take one of the pieces of wool (equal pieces). Share this with five people (don't forget yourself).

1. How long is each share?

Get a shorter piece of ribbon.

2. How many pieces of ribbon of this length can you cut from the longer piece?

C (coins) - MONEY

Make one pile of money with: PILE A { two quarters, four dimes, five nickels, seventeen pennies }

Make another pile with: three quarters, two dimes, four nickels, twelve pennies } PILE B

1. How much does PILE A have?
2. How much does PILE B have?
3. How much do both piles have altogether?
4. Which pile has the greatest amount? How much more does it have?
5. What do you do to find twice the amount as PILE A? How much is that amount?
6. How much is three times the amount of PILE B?

Make two piles of pennies that are not equal.

1. Add the two piles together.
2. Subtract the two piles.
3. What must be added to the smallest pile so that both piles will be equal?
4. What do you do to find twice the amount?
5. What do you do to find three times the amount?
6. Share the pennies among three people. How many pennies will each person get?
7. Share the pennies with five people. How much will each person get?

There are some sums of money written below.

Use a box of coins to find out how few coins can be used to make each amount.

For example, 56¢ is one half-dollar, one nickel and one penny.

1. 19¢
2. 41¢
3. 75¢
4. 91¢
5. 97¢

Make three little piles of coins.

1. Count the amount of money in each pile and write the three answers in your notebook.
2. Now make up a number sentence to find out how much money there is altogether in the three piles.
3. See how many other adding number sentences you can make using the three piles of coins.

Using the coins find out how many different ways you can make $1.00. Record.

e.g. (25¢) + (25¢) + (25¢) + (25¢) = $1.00.

Buy five articles that are on sale. Add to find how much they are. Buy three articles your mom would use for supper. Add. Record in Math. book.

---

1. Look through the toy section of the catalogue. Buy some things you would like for your birthday. Add up how much it will cost. Record in Math. book.

How much change do you get if you spend:

a. 7¢ out of a dime
b. 6¢ out of a quarter
c. 37 ¢ out of a half dollar
d. 23¢ out of a half dollar
e. 68¢ out of 3 quarters

Show your answers in this way:

a. 13¢ out of a quarter

(1¢)	(1¢)	(5¢)	(5¢)
14	15	20	25

or

(1¢)	(1¢)	(10¢)
14	15	25

M - MEASUREMENT

Find some things to measure.
Choose one thing which is long.
Choose another thing which is short.
Measure how high something is.
Measure the distance round something.

1. When you have finished measuring, write your answers in your notebook in a good sentence.

How high is the wall? Use the part of the wall that is jutting out near the door.

There is a clue on the back BUT don't use it unless you get stuck. (bricks)

1. In your notebook, explain how you found out how high the wall is.

Find a partner.

1. Draw six straight lines with your ruler in his notebook. Let your partner do the same in your book. Each line must have a different length. Each line must measure only in inches and half inches.

2. Now you have to measure the lines in your notebook.

3. Write the length near each line.

Make strips of paper which are five inches long.

1. How many of these strips will cover a foot ruler?
2. How did you find that out?
3. How many of those strips will cover two foot rulers?
4. Explain how you got that.
5. How many strips will cover a yard?

---

1. How long do you think the bulletin board by the door is? Put your guess in feet and inches.

2. After measuring it in feet and inches look at both answers. Which had the greater number of feet and inches? How much was the difference?

Find one window screen in the room.

1. Guess what the length is.
2. Guess what the width is.
3. Find out how long the two parts are to the nearest inch.
4. What is the difference between your guess of the length and the real length?
5. What is the difference between your guess of the width and the real width?

---

1. How wide do you think the cupboard door is?
2. Measure it to find out how many inches it is.
3. How close was your guess?

Write down in your notebook what you did, and what you found out

---

Find two books which have different lengths.

1. Measure with your rubber how many rubbers long the two books are.
2. Which book is longer? How much longer is it? Find out the answer by putting it in a number sentence.
3. Now do the same thing with the width of each book.

---

1. Measure the height of your partner. Put your answer down to the nearest inch.
2. Now have your partner measure you.
3. Are you taller or smaller than your partner? Find out the difference between your heights by making a number sentence.

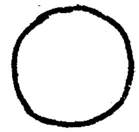
---

1. Write down the names of six things in the classroom which are round in shape.
2. Opposite the name of each one, write your guess as to how wide it is at its widest point.
3. Now take these things and draw round each one on a sheet of paper. Cut out each circle. Fold each circle exactly in half.
4. Now measure the crease line.

←---crease line

5. How close was your guess to the right measurement?

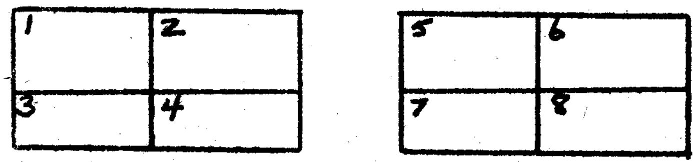
Sh - SHAPES

What is the name of this shape? 

Find some things in our classroom or outside that are this shape.

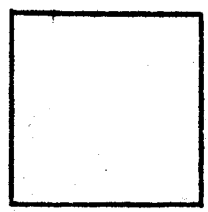
Fold a large piece of paper like this:

Front Back



In each box make a picture of some things that have a

 shape.

What is the name of this shape? 

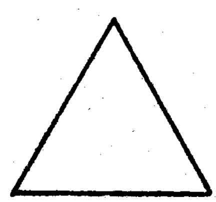
Find some things in our classroom that are this shape.

Fold a large piece of paper like this!

Front Back

In each box make a picture of some things that have a

 shape.

What is the name of this shape? 

Find some things in our classroom or outside that are like this shape.

Fold a large paper like this:

Front Back

1. Using 26 gummed squares and experience chart paper, make as many squares and rectahgles (different in shape) as you can.

2. Each shape must have 26 squares.

3. What happens to the width and length of each figure?

4. What happens to the area?

5. Multiply the length and width of each figure and see what you get.

---

1. Using 24 squares each time makes as many different rectangles as you can. Record the length and width of each.

S - SUBTRACTING

Take two cups and some dried peas. Put some of the peas in one cup and the rest in the other cup.

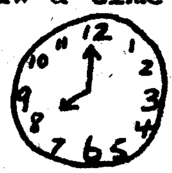
1. Find how many peas there are in each cup.
2. How many are there altogether? Write it down in two ways.
3. Which cup has the greater number of peas? Put your answer in a number sentence.

1. Draw a large clock face in your notebook.
2. Around the outside, write in the hours from one to twelve.
3. Just inside the circle, and opposite the numbers you have written, write in the Roman numerals from I to XII.

   You may like to know that the numbers we always use, like 1, 2, 3, 4 and so on, are called Arabic numerals.
4. Why would they be called that?

---

1. Draw six clock faces in your book.
2. On each face draw a time when something important happens during the day.

   Like this: 

   At eight o'clock I have my breakfast.

   (Don't forget to write down what happens at that time.)

---

1. Use a stop watch to find out how long you take to count a hundred beans. Be as quick as you can. Now your partner will do the same

   Now you have finished counting. Put away the watch and the beans.
2. Next, work out how long it would tak e you to count 500 beans. Do this in your notebook.
3. Now find out how long you would take to count 700 beans.

Look through the pages of a calendar to find six important days during a year.

1. Write down the names of the special days and their dates.

   Sometimes dates are written this way: 14.1.49 -- This means the fourteenth day of January in the year of 1949.
2. Write out these dates in full:

21.7.62	1.10.54
17.11.59	9.2.71

3. Write down you birthday in at least two ways.

---

1. If there are 30 days in November, how many weeks will there be in that month?
2. If there are 31 days in August, how many weeks will there be in that month?

   Use the calendar to see if you are correct.

HOW TO MAKE A PENDULUM:

1. Roll some plasticine into a ball.
2. Cut off 5 or 6 feet of string.
3. Make a small loop at each end of the string.
4. Put one loop round the ball and put the string right into the plasticine so that it can't come out.
5. Hang the other end of the pendulum on a nail.

AN EXPERIMENT WITH YOUR PENDULUM

1. Swing your pendulum so that it does not bump into anything.
2. Count how many times a pendulum goes from one side to the other in one minute.

   Use a clock with a second hand or have your friend count slowly to 60.
3. Write down the number of times your pendulum was swinging.
4. Try it again.

   Was the number the same? Why? or Why not?

DIFFERENT PENDULUMS

1. Stick more plasticine on the ball of your pendulum.
2. Now count the number of swings in one minute.
   Use your notebook to put down what you learned.
3. Does the extra weight make any difference?
   Why?
4. Try the same thing again.
5. Now make the ball of plasticine smaller.
   How many times does it swing in one minute?

DIFFERENT LENGTHS OF STRING

Make the string shorter.

1. How many times does it swing in one minute?

Now make the plasticine ball bigger.

2. How many times does it swing in one minute?

Make the plasticine ball smaller.

3. How many times does it swing in one minute?
4. If you need more practice doe the experiment a few more times.
5. What did you find out by doing this experiment?

maybe too hard

REVIEW

Does the number of swings a pendulum makes in one minute depend on:

1. The length of the string?
2. The weight at the end?
3. How hard you swing it?

MAKING A CHART

1. Use squared paper to make a chart like this:

Length of String	Number of Swings in One Minute
6 feet	
5 feet	
4 feet	
3 feet	
2 feet	
1 foot	

2. Now fill in the chart with what you found out.

YOUR HEART BEAT

1. Make a pendulum which swings once a second.
   It will need a 39 inch string.
2. You will need 2 friends to help you.
3. While one person counts 60 swings of the pendulum, you count the pulse beat of your other friend.
4. Make a list of the results for different people.

TIME

- stop watch

What can you do in two minutes?

Estimate the number of times and try it.

Outside
- bounce a ball
- say the alphabet
- jump up and down
- hop on one foot
- skip
- clap your hands

Inside
- tap foot
- How far can you count?
- Write your name neatly.
- How many words can you make from: SANDWICH?

/ RECORD.

TIME

1. Make a graph about how many hours you were in bed last night.
2. Ask five other people. Ccmpare and tell your story.

V - VOLUME

Find out how much water rises when you put each stone in it.

You need a jam jar and a ruler.

Put enough water in the jar to cover the largest stone. Measure how deep the water is. Add water if you need to, to make it an easy measurement like 2 inches or 2½ inches.

Put stone 1 in the water and measure how high the water went.

Do this with all the stones.

Make a list of what you found out.

Fill the bottle with water.

1. Guess how many equal cups of water you will get out of the bottle.
2. Now find out how many you will get out of the bottle.
3. How will you make sure that you will put the same amount of water in the cups every time?

W - WEIGHT

Weigh two things together. Now weigh one of them by itself. Find the weight of the other thing without weighing it.

1. Show how you have done this by writing it as a sum in your book.
2. Now do this six more times, using different pairs of things each time.

Which is heavier, a cup of peas or a cup of rice?

1. Guess first.
2. Now use your scales to find out.

Which cup has the greatest amount of peas?

1. Guess first by just looking at the cups.
2. Now weigh them to see if you are correct.

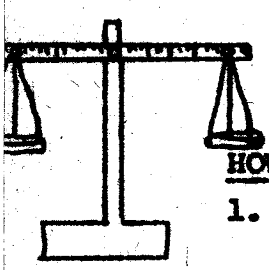
   Put down what you found out in your notebook in your own way.

---

1. Take the jar of dried peas and guess how many two-ounce bags you can get out of it.

   Take a little bag and put two ounces of peas in it. Use the scale to help you. Keep going until you have used all the peas in the jar.
2. How many two-ounce jars did you fill? How close was your guess?

MATERIALS NEEDED: an old ruler
some inch nails
string or a stand
two large tin lids both the same size
hammer
tool to put holes in ruler



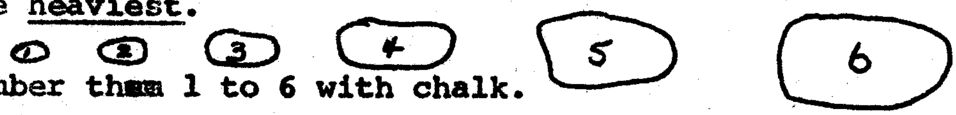
HOW TO DO IT:

1. First make three small holes in the ruler on the one-inch, six-inch and eleven-inch lines.
   Be careful you do not split it.
2. Then make three holes in the rim of each tin lid. A hammer and a nail is an easy way of doing this.
   Make the holes the same distance apart.
3. Put a one-inch nail through the centre hole of the ruler.
4. Put string (double) through the holes in the lids and put the string through the holes at the one-inch and eleven-inch marks.
5. At the six-inch mark, attach the balance to a stand.

6. If the ruler doesn't hang level, stick a small lump of plasticine on the side of the ruler which is higher. Move the lump along until the scales are level.

---

1. Find six stones small enough to go into a jam jar, but not too small.
2. Weigh them in your hand and put them down from the lightest to the heaviest.



3. Number them 1 to 6 with chalk.
4. Now, using your scales, weigh the stones one against the other.

   Keep doing this until you have found the heaviest.

   Then find the next heaviest and so on, until you have done them all and have put them in order.
5. Are they still in the same order as they were when you guessed the weight?

   If not, rub out the chalk numbers and put the right numbers down. Put them in your jar.
6. Now, give your stones to your partner who will find out if you did the right thing.

---

1. Get a long piece of wood and a saw.
2. Without measuring, saw the wood into six different-sized pieces.
3. Weigh each piece in your hand and try to put the pieces in order with the lightest first.
4. Number each piece with chalk.

5. Next weigh one against the others in the scales and put them in order of weight.
6. Are they still in the same order? If not, then rub out the chalk and put the correct numeral down.
7. Have your partner check the weight of your blocks to see if they are correct.

---

hese weights have been made if ou use the first card on eight-pg. 3

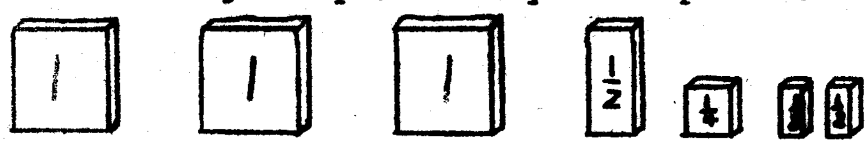
Use your weights to weigh six things which you find in the classroom.

1. In your notebook make a list of the things you weighed and their weights.

Objects	Weight in Wuzes
1. three pennies	¼ Wuz
2. one box of elastics	1¼ Wuzes

1. Make four cubes of plasticine with sides about 1 inch long.
2. Weigh them against each other in the scales. Add pieces of plasticine until they are all the same weight.
3. Cut one cube in half.
4. Weigh the two halves against each other and take plasticine from one piece to the other piece if necessary. Make sure they are of equal weight. 
5. Weigh the two halves against a whole cube to make sure the weights are still equal. 
6. Make up your own name for the cube, say 1 Wuz.

   The halves would then be ½ Wuz.
7. Cut one half in two. 
   Weigh it against the other part to make sure it is equal. What would you call these two smaller parts?
8. Then take one of those parts and cut it in half. Weigh these two smallest parts against each other.
   What name would you give to them?
9. Take something sharp so that you can put the weight on each part.



Weigh the six stones that you used on card ___.
Use your weights.
Make a list of what you found out.

Stone	Weight in Wuzes
1	
2	
3	
4	
5	
6	

Find the five packages that are marked A,B,C,D,E. Weigh package A on the scales. Now guess how much the other packages weigh by holding each one in your hand.

1. Write down your guesses in your notebook.
2. Use scales to weigh packages B,C,D,E. Put your answer beside your guess.
3. Make a number sentence to see how close your guess was.

Compare the weights of 1 cup of flour, sugar, rice, macaroni, water, and salt to washers. Try to estimate which is lighter or heavier. /. Record your answer by way of a bar graph or a pictoral graph.

Here is a ball of plasticine.
Estimate which of the objects on your table weigh the same.
Now weigh them to see how close you were.
/. Record.

---

1. Pick two objects from the table. Using your hands, estimate which is heavier.
2. Using a book from your desk, find 5 other objects in the room which are:
   a. lighter than the book
   b. heavier than the book
   c. the same as the book
3. From the room, choose your own object that you can carry. Find things in the classroom that are:
   a. lighter than your object
   b. heavier than your object
4. You may use the balance scales to prove anything you are not sure of.
5. Record answers to numbers 1, 2, and 3 in an interesting way.

<u>THINGS YOU WILL NEED:</u> <u>5</u> crayons
a box of counters
a set of balancing scales

<u>WHAT TO DO:</u>

1. Into one pan put <u>5</u> crayons. How many counters do you think you will need to balance the pans? Write down your guess.
2. Now put the counters, one at a time, into the other pan until the pans balance.

<u>WHAT DID YOU FIND OUT?</u>

/. How many counters did you need? Did you guess the right number of counters?

7. Examples of "Applied" Mathematics Problems for the High School Curriculum

There are several considerations in deciding which applications lead to meaningful learning in mathematics. Pollak lists the following in one of the chapters in Mathematics Education:

1. Underlying assumptions:
   a) Applications are essential for honest and complete picture of mathematics.
   b) Applications are important motivational material.
   c) Logical and well-reasoned discussion will shed light on a solution to the problem.
   d) The essential goal is to have students experience the process of model-building.
2. The majority of "applied" problems in texts today involve only translating to mathematical terms and then applying a standard method to get a solution. The text problems should be more like actual applications where it is unclear how to derive a problem (or model) from a particular situation. It is crucial that the applications be honest: there should be a relationship between the mathematical model and the real life situation which is clearly understood.
3. There are three general areas of applied mathematics:
   a) everyday life
   b) some discipline other than mathematics
   c) some other branch of mathematics

   Pollak considers only the first two of these in this paper. Almost all major fields of human endeavour and many everyday situations lead to significant applications of mathematics.
4. Types of applied mathematics Pollak considers are those involving setting up mathematical models, games and puzzles, and experiments and data collection. The last type is particularly well-suited for the elementary level but causes problems at the secondary level. At the secondary level there is a need for a logical sequence to the topics studied. The "natural order" of the topics in mathematics may not agree with the "natural order" of motivational scientific topics.

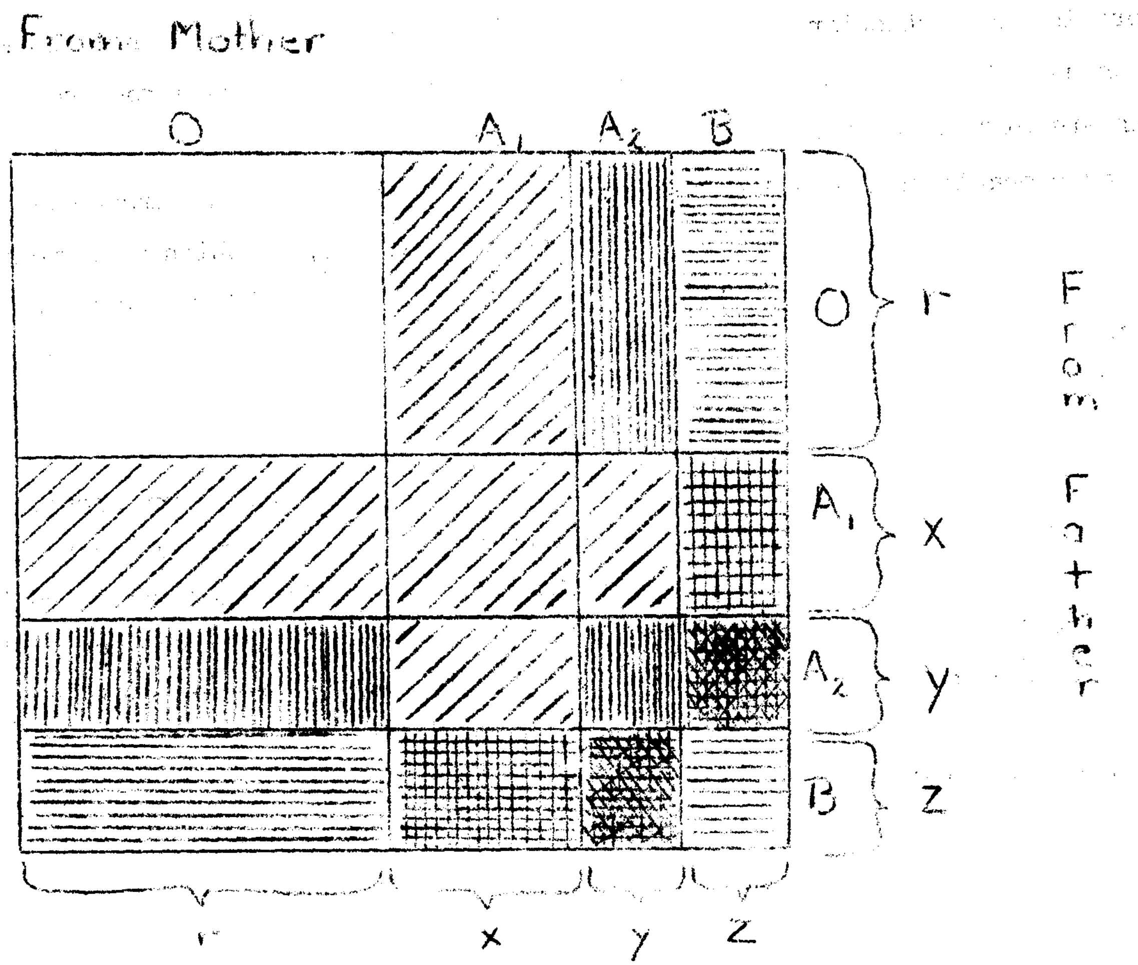
   "If one really tries to integrate science and mathematics teaching, he risks the danger of interfering with the development of at least one of the two partners."

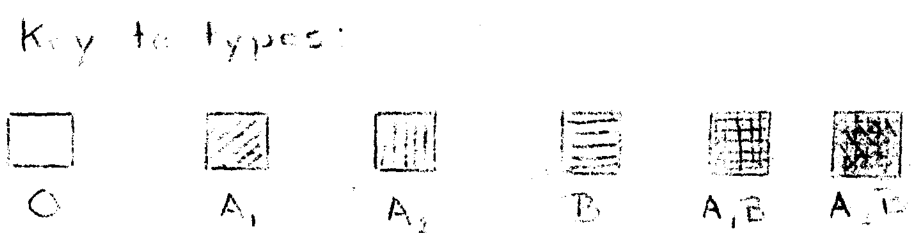
5. The following stages are involved in applications:
   a) recognition that something needs exploring in order to gain better understanding of a situation;
   b) formulation of a precise mathematical model. It must be complicated enough to be an honest representation and yet simple enough to have some chance of solving it. A successful model should not be greatly affected by small changes in the basic assumptions. Any consequences that seem intuitively wrong should lead to improving the model.
   c) Obtaining a solution;
   d) relating results and new understanding to the original situation.
6. The aim of this approach is to give the students the experience of discovering math for themselves. For this it is necessary to understand <u>when</u>, <u>how</u>, and <u>why</u> the mathematics work.
7. Evaluation is based on the student's understanding of the original situation; has his understanding increased, and is he better able to make predictions?

(A) An application of systems of quadratic equations to human heredity (blood groups)

--see W. W. Sawyer, <u>The Search for Pattern</u>, pp. 274-78

In this problem we will consider only the ABO blood groups. In determining the blood type of a child there are four possibilities for the father's contribution (O, A_1, A_2, or B) and four possibilities for the mother's contribution (the same four).





From the diagram above we can see how the genes combine. If a child gets an A_1 gene from one parent and an O gene from the other parent his blood type will be A_1. If he gets an A_2 gene and an O gene his type will be A_2, and if he gets a B gene and an O gene his type will be B. But if he gets an A_1 gene from one parent and a B gene from the other his blood type will be A_1B. Likewise an A_2 gene and a B gene combine as type A_2B. Therefore, from testing the child's blood, one cannot determine exactly which genes he received from his parents in some cases. The problem is then to find, from a sample of children, the percentages of each gene in the parent population.

Here is one particular case: starting with a sample of 10,000 children we will then have 100 rows and 100 columns in our diagram. Assuming a row and a column are picked at random, any point of intersection of a row and a column will be as probable as any other point. Our problem is to find how many rows or columns there should be for each type gene: O, A_1, A_2, B, which we assign the variables r, x, y, z, respectively. In our sample of 10,000 we have the following: 4356 type O; 3507 type A_1; 973 type A_2; 828 type B; 252 type A_1B; and 84 type A_2B.

The type O blood has only one region on the diagram and that corresponds to r^2 places. And we know from the sample that those r^2 places equal 4356. Type A_1 blood has five regions: two rx regions, two xy regions and one x^2 region. For type A_2 we have two ry regions and one y^2 region; type B: two rz regions and one z^2 region; type A_1B: two xz regions; type A_2B: two yz regions.

So we get the following system of equations:

(1) $r^2 = 4356$

(2) $x^2 + 2rx + 2xy = 3507$

(3) $y^2 + 2ry = 973$

(4) $z^2 + 2rz = 828$

(5) $2xz = 252$

(6) $2yz = 84$

These equations may be solved as follows:

(1) $r^2 = 4356$

$r = \sqrt{4356}$

$\boxed{r = 66}$

Then notice equation (3) is almost a perfect square. By adding r^2 to both sides of the equation we can make it one.

(3) $y^2 + 2ry = 973$

$y^2 + 2ry + r^2 = 973 + r^2$

$y^2 + 2ry + r^2 = 973 + 4356 = 5329$

$(y+r)^2 = 5329$

$y + r = 73$

$\boxed{y = 7}$

Solving equation 4, in the same way we get z=6

Now substitute z = 6 into equation 5)

5) 2xz = 252

12x = 252

x = 21

Checking these values in the remaining two equations we find that they work. Therefore we arrive at the following percentages:

type O: 66%; type A_1: 21%; type A_2: 7%; and type B: 6%

(B) Application of systems of equations to circuits and resistances.

- from W. W. Sawyer, The Search for Pattern, pp. 133-38

In order to solve a problem of this type we need to remember two things: Ohm's Law and the principle involved at junctions of circuits. Ohm's Law states that the voltage in a system is equal to the product of the current, in amperes, and the resistance (V = AR or A =V/R). The principle is that at any junction in a circuit the current flowing into the junction is the same as the current flowing out. Therefore, in the diagram below x = y + z where x, y and z are measures of the current in amperes.

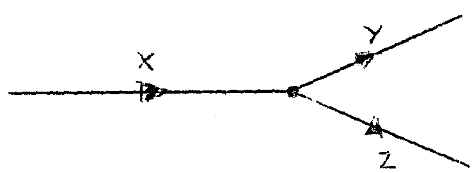


Figure 1.

Now we can start on a problem.

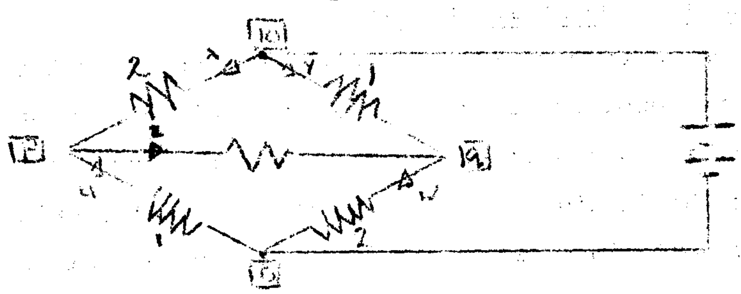


Figure 2.

In figure 2, x, y, z, u, and w are currents measured in the directions shown by the arrows. The numbers by the WWW symbol represent the size of the resistance in ohms. The numbers in the boxes (10, 0, p, q) represent the potentials at each junction in volts. Using Ohm's Law we have 5 equations:

x = (10 - p)/2 or x = 5 - 1/2p (1)

y = (10 - q)/1 or y = 10 - q (2)

z = (p - q)/2 or z = 1/2p - 1/2 q (3)

u = (p - 0)/1 or u = p (4)

w = (q - 0)/2 or w = 1/2 q (5)

From the junction principle we get two more equations:

$x = u + z$ (6)

$y + z = w$ (7)

Now we have seven equations in seven unknowns. This can be reduced to two equations in two unknowns by substituting equations (1) - (5) into equations (6) and (7)

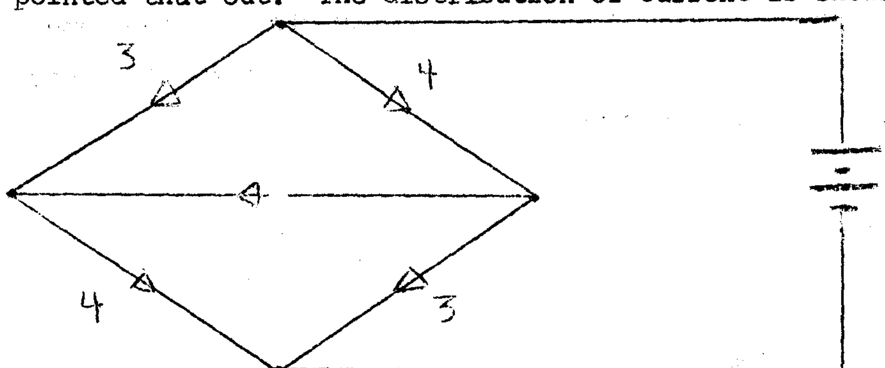
$5 - 1/2p = p + 1/2p - 1/2q$ or $10 = 4p - q$ (8)

$10 - q + 1/2p - 1/2q = 1/2q$ or $20 = -p + 4q$ (9)

Solving (8) and (9) gives us $p = 4$ and $q = 6$. Putting these values into (1) - (5) yeilds:

$x = 3, y = 4, z = -1, u = 4, w = 3$

Notice that $z = -1$. This means that the arrow in figure 2 was drawn in the wrong direction. But that doesn't matter because the algebra of the problem pointed that out. The distribution of current is shown in figure 3.



(C) A Concrete approach to the quadratic Function

1. The students are asked to perform two experiments and plot their results. One of the experiments could involve a piece of blotting paper with its bottom tip in a beaker; it acts as a wick and the students would measure how far the liquid has soaked up the paper at regular short time intervals. They would then graph distance vs. time and discuss what happens in between the time intervals that are measured. The students would probably join the points to form a smooth curve, and a curve closely resembling a parabola should result. The parabola would be a "horizontal" or a "vertical" one depending on how the student labels his axes. A second experiment could be one involving the horizontal metronome (see p. 91-98, Mathematics Through Science, Part III, SMSG).

2. The students are asked to find examples of parabolas in everyday life - e.g., the cross-section of the radiotelescope and car head-light, arches in a chapel and of bridges, geometric patterns (curve stit-ching), relationships such as the stopping distance of a car when plotted against time, etc.

3. The students are led to discover the form of the equation of a vertical parabola (quadratic function): $y=x^2$, $y=1/2x^2$, etc.

4. The students solve proportion problems involving one variable being proportional to the square of another (e.g., length of a square vs. area, energy vs. velocity, etc.). He is led to the result that to check whether a curve is parabolic you plot y^2 versus x^2 and the result will be a straight line.

5. The students learn (perhaps through group projects of one or two days) that the Greeks considered the conic sections, and that the parabola is one of those conic sections with the property that the distance from a point on the parabola to a fixed point called the focus always equals the distance from that point to a fixed line called the directrix. They also relate this to the property that if a light source is put at the focus of the parabola, the rays will reflect in such a way that parallel rays of light will emanate from the parabola (e.g., a searchlight). If you cut a hard-boiled egg, what types of curves result?

6. The students discover the properties of the parabola in the form $y = a(x-p)^2+q$ (preferably using the vector translation notation). Depending on their ability, students would investigate one or more of the following:

- is a freely hanging chain a parabola? (No, it's a catenary, whose equation is $y = 2^x + 1/2x$)
- does water coming out of a small hole near the bottom of a large can full of water follow a parabolic path?
- use a long inclined plane with small cars or marbles. Plot distance vs. time. A parabola? Find the equation.
- use data for the planets for Kepler's third law and plot time vs. length of axis. A parabola? If so, find equation. If not, can you suggest another relationship? (See Eves, p. 272. The relationshi

- ask students to discover the equations of parabolic arches of neighbouring bridges. They may have to take a photograph; determine a convenient axis system, etc. If there are none in the neighbourhood, find some large photographs. Conversely (but not as good), give the equations describing some famous bridges and ask the students to plot them (see p. 119 in Mathematician's Delight, W. W. Sawyer).

7. Introduce the example of a farmer having a certain amount of fencing to enclose a field of largest area, with one side of the field being a wall and not needing to be fenced. By making a graph, the students can solve the problem, but will find it awkward and lengthy. The student learns the method of completing the square (first using geometric diagrams, so that he realizes what he is doing). The student solves problems using this method such as "A rain gutter, open at the top and rectangular in shape, is to be made from roofing tin that is 12 inches wide. How should the tin be bent to produce a gutter with greatest carrying capacity?". The students may suggest various methods of attack beside the method of completing the square.

8. The better students would perform some experiments in order to discover the need for the inverse of the quadratic function (e.g., the oscillating spring experiment in Mathematics Through Science, Part III, p. 104-114).

9. There would be additional problems and investigations provided for the better student. For example:

- investigation of difference patterns, leading up to arithmetic progressions and diagonals (see The Search for Pattern, W. W. Sawyer)
- finding the equation of a parabola given three points on the curve (e.g., (-1,13), (1,5), and (2,7) lead to $y=2x^2-4x+7$) The student can investigate whether it is possible to fit a curve through of the form $x = ay^2 + by + c$. He then is asked to find the equation of a parabola through a set of statistical points, and he will have to approximate the curve, since it may not go through all the points exactly, and find a method for finding the equation.

- do the investigation suggested in the General Motors Newsletter <u>Parabolas and Automotive Safety</u> (available from GM in Oshawa)
- investigate the difference between the properties of the parabola, hyperbola, and ellipse (e.g., see Vergara, p. 182-4: where do shooting stars come from?).

10. A section on the quadratic equation would follow, showing some of the ways in which these equations were solved in history, (e.g., the Babylonians, Pythagoreans, Hindus, and Arabs).

8. The Next Steps

The seminar participants hope to implement some of the ideas developed during the seminar during the 1971-72 school year. Evaluations of such experiments will be sent to the seminar co-ordinator and distributed among the participants. It is our hope that such experiments and more thought about the basic issues and direction will set the stage for a more comprehensive workshop in the area of mathematics during the summer of 1972. It is our hope that such a workshop will be able to write some actual units in certain areas.

To all readers who did not participate in the seminar we direct the following request: please realize that our work is tentative, and the only reason we are making the report available to non-participants is to give you food for thought in the design of a mathematics curriculum and a few concrete ideas. Feel free to criticize and give us your comments on any aspect of the report and if you do try out any of the activities, we would appreciate receiving your reactions. They can be mailed to Harro Van Brummelen, 8020-160 Street, Edmonton, Alberta.

APPENDIX

This appendix contains lists of references that were prepared on certain topics by the seminar participants. Included are those books that were found most helpful in the particular area the person concerned was working on.

I FOUNDATIONS IN HISTORY

General

Dentzig, Tobias. Number -- the Language of Science. (Doubleday Anchor Book A67) Doubleday & Company, Inc., Garden City, New York; 1954. (originally by Macmillan; 1930)

Kline, Morris (ed.). Mathematics in the Modern World. Readings from Scientific American. W. M. Freedman and Company, San Francisco & London, 1968

Newman, J. (ed.), The World of Mathematics. Simon and Schuster; New York, 1956. (Vol. 1 - Vol. 3 - Vol. 4: p. 2317 - 18 & whole article).

Sawyer, W. W., A Path to Modern Mathematics. Penguin Books, Harmondsworth, Middlesex, England, 1966.

Foundations (books)

Barker, Stephen F., Philosophy of Mathematics. (Prentice-Hall Foundations of Philosophy Series) Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964.

Benacerraf and Putnam, eds. Philosophy of Mathematics. Selected Readings. Prentice-Hall.

Beth, Evert W., The Foundations of Mathematics. Harper Torchbook. (difficult - much symbolic logic)

Burtt, E.A., The Metaphysical Foundations of Modern Science. Anchor paperback.

Cantor, G., Transfinite Numbers. Dover, New York; reprint of 1915 translation.

Cassirer, Ernst, The Problem of Knowledge. Yale University Press, New Haven, 1950.

Cassirer, Ernst, Substance and Function. Dover paperback (esp. chapters 2 & 3).

Dedekind, Richard, Essays on the Theory of Numbers. Dover paperback.

Fraenkel & Bar-Hillel, Foundations of Set Theory. (North Holland?, Humanities Press? -- out of print -- excellent book.

Frege, Georg, The Foundations of Arithmetic. Harper Torchbook.

Jammer, Max, Concepts of Space. Harvard University Press.

Kornner, S., The Philosophy of Mathematics. (Harper Torchbook TB547H) Harper and Row, Publishers, New York, 1962 (originally by Hutchinson & Company, Limited, London, 1960).

Poincart, Henri. 4 Vols. on science. The Value of Science; Science and Hypothesis; Science and Method; Mathematics and Science: Last Essays. Dover paperbacks.

Renyi, Alfred, Dialogues on Mathematics. Holden-Day, San Francisco, 1967.

Russell, Bertrand, Foundations of Geometry. Dover paperback.

_______________, Introduction to Mathematical Philosophy. Humanities Press.

_______________, Principles of Mathematics. The Norton Library (paperback).

Spier, J.M., An Introduction to Christian Philosophy. (University Series--Philosophical Studies) Craig Press, Nutley, N.J., 1966.

Weismann, Friedrick, Introduction to Mathematical Thinking. Harper Torchbook (Now available only in hard cover, I believe)

Wezl, Hermann, Philosophy of Mathematics and Natural Science, Princeton Hardback.

Foundations (papers/articles)

Hammer, P. C. "The Role & Nature of Mathematics", The Mathematics Teacher: Vol. 57; Dec., 1964; pp. 514-521.

Hempel, C. G., "On the Nature of Mathematical Truth", American Math Monthly: Vol. 52, Dec., 1945; pp. 543-556. (Also in World of Mathematics--Newman, ed.)

Kuyk, Willem, "The Irreducibility of the Number Concept", Philosophia Reformata pp. 37-50. (1966)

Strauss, D.F.M., "Number-Concept and Number-Idea" Philosophia Reformata, Vol. 35, #3-4, pp. 156-177, 1970.

Van Brummelen, Harro, "The Place of Mathematics in the Curriculum", paper to summer seminar in education -- math; OACS/AACS, 1971.

History

Boyer, Carl B., The History of Calculus; Dover

Cassiere, Ernst, The Problem of Knowledge, Vol. 4., Philosophy, Science, and History since Hegel. Yale paperback; esp. first section.

_____________, Substance and Function. Dover paperback, esp. chapters 2 & 3.

Eves, An Introduction to the History of Mathematics (many historical problems)

Klein, Felix, Elementary Mathematics from an Advanced Standpoint, 2 Vols. Arithmetic, Algebra, and Analysis and Geometry. Dover paperbacks.

Kline, Morris, Mathematics in Western Culture. Oxford-Galaxy Book.

Kuhn, Thomas, The Structure of Scientific Revolutions. The University of Chicago Press, Chicago, 1962.

Neugebauer, Otto, The Exact Sciences in Antiquity, (Harper Torchbook TB 552) Harper & Brothers, New York, 1962. (Hardcover edition 1952, Princeton University Press).

Struik, Dick J., A Concise History of Mathematics. Dover paperback.

Van der Waerden, Science Awakening, ???

2. THE LEARNING OF MATHEMATICS

Association of Teachers of Mathematics Handbook, Some Lessons in Mathematics (Cambridge University Press)

Ausubel, D., and Robinson, F., School Learning (Toronto: Holt, Rinehart and Winston, 1969)

Avital, S., and Shettlesworth, S., Objectives for Mathematics Learning (Toronto: Ontario Institute for Studies in Education)

Brackenbury et al., Rational Planning in Curriculum and Instruction. (Washington: National Education Association, 1967)

Cambridge Conference Report, Goals for School Mathematics, (Houghton Mifflin, 1963).

DeGraaff, Arnold H., The Nature and Aim of Christian Education, (Toronto: AACS, 1970).

DeGraaff, Arnold H., The Educational Ministry of the Church (dissertation published in 1967. Available from Tomorrow's Book Club in Toronto).

Henderson, Teaching Secondary School Mathematics, (#9 of the series "What Research Says to the Teacher", N.E.A., 1970).

National Society for Study in Education. Mathematics Education. Chapters include: Psychology and Mathematics Education, issues in the teaching of mathematics, historical background of innovations in mathematics curricula, applications in mathematics, etc. A good overview. (Chicago: The University of Chicago Press for NSSE).

Oppewal, Donald, Toward a Distinctive Curriculum for Christian Education (From the Reformed Journal, Sept., 1957; available from the AACS).

Tyler, R., Basic Principles of Curriculum and Instruction. (Chicago; the University of Chicago Press, 1950).

Woltersdorff, N., Curriculum, By What Standard? (Grand Rapids: National Union of Christian Schools, 1966).

3. RESEARCH BOOKS FOR MATH ACTIVITIES IN THE PRIMARY GRADES

Bates, Brown, Grodzinski, Luftman, Mathaction Series; Copp Clark (517 Wellington Street W., Toronto 2B). Good source for games and activity cards; over-emphasis on logic: price - $2.50 per book.

Project Mathematics (Series 1-6) Holt, Rinehart, and Winston of Canada Ltd., (Toronto), 1971. Interesting workbooks, but somewhat abstract and lacking in continuity.

P. W. Cordon, Mathematics and Measuring. Series: Measuring Length, Weight, Time, Quantity, Area, and Volume. MacMillan and Co. Ltd., (70 Bond Street, Toronto 2), 1968. British publication; excellent directions for individual discovery-type activities; a basic set for Grades 3 - 6, highly recommended.

Spitzer, Herbert F., Enrichment Activities for Grade 4. McGraw-Hill Book Co., Toronto, 1964. Interesting games, puzzles, and number tricks.

R. A. J. Pethen, Workshop Approach to Mathematics. MacMillan Co. of Canada Ltd., Toronto, 1968. Teacher's Guide accompanies individual activity cards; some good ideas, but too difficult words for the primary grades. Sample set recommended.

Mathematics in Primary Schools, Curriculum Bulletin No. 1, London: Her Majesty's Stationery Office, 1969. Basic guide and source book for the discovery approach of math learning; essential in implementing such a program. also:
E. E. Biggs, Freedom to Learn; Addison-Wesley (Can.) Ltd.; an application of British methods to Ontario schools.

Nuffield Mathematics Project. I do, and I Understand Mathematics Begins Computation and Structure. Nuffield Foundation; Chambers and Murray, Newgate Press Ltd.,London, 1967. (Publ.). Similar in approach to Math in Primary Schools.

Brydegaard and Inskeep (Edits.), Readings in Geometry from the "Arithmetic Teacher"; National Council of Teachers of Mathematics, Washington, D.C., 1970. Articles for teachers, including illustrations and suggested activities for the primary grades; worthwhile.

Smith, Thyra, The Story of Measurement and The Story of Numbers, Basil, Blackwell & Mott Ltd., Oxford, 1959. Excellent history readings for junior level (Gr. 4-6). Attractive format, development up to present day.

Boyce, E.R., How Things Began - Arithmetic (Book 2); MacMillan and Co., 1[illegible]4. Vocabulary geared to Grade 3 and 4; describes development of numerals in ancient times; includes questions at the end of each chapter.

Jonas, Arthur, Archimedes and His Wonderful Discoveries, Prentice Hall Inc., (Englewood Cliffs, N.J.), 1963. Interesting style and illustrations for Grades 4-6.

Smith, David Eugene, Number Stories of Long Ago; National Council of Teachers of Mathematics, (1201 Sixteenth St., N.W., Washington, D.C., 20036). The history of math in story form using imaginary characters; first edition in 1919; now in new attractive format for Grade 4 and up.

Razell and Watts, Circles and Curves, Symmetry, and Four and the Shape of Four Rupert Hart-Davis; London, 1967. (General Publ. Co., 30 Leaside Road, Don Mills, Ontario). Well illustrated booklets with relevant geometry material for Grades 4 - 6; price $1.10; a worthwhile series for the classroom.

A History of Measurement; Ford, Educational Affairs Dept. The American Road, Dearborn, Michigan. (or public relations department, Oakville, Ontario). Poster with large colourful pictures about the history and use of units of linear measurement; for Grades 3 and up.

Gale, D. H., The Teaching of Numbers; Hulton Educ. Publications; (Bellhaven House Ltd., 1145 Bellamy Rd., Scarborough, Ont.), 1963. A basic approach to the teaching of number for the teacher's use.

Bates, Irwin, Hamilton, Developmental Math Cards; Addison-Wesley (Can.) Ltd., (57 Gervais Drive, Don Mills, Ontario), 1969. Glossy colourful cards, accompanied by teachers' guide; attractive to primary grades, but some vocabulary too difficult.

Adler, Irving, The Giant Golden Book of Mathematics, Exploring the World of Numbers and Space; Golden Press, New York, 1966. Large, hardcover, book geared for Grade 4 and up; suitable material for activity cards.

Kennedy, Leonard M., Models on Math in the Elementary School: Wadsworth Publ. Co., Inc., Belmont, California, 1967. Describes how to make learning aids; includes appendix of companies that manufacture learning aids.

Weiss, Irwin, Zero to Zillions, Scholastic Book Services, New York/Toronto, 1966. Booklet of number games and activities in children's language.

Kidd, Myers, and Cilley, The Laboratory Approach to Mathematics. Chicago, Science Research Associates, 1970.

4. ENRICHMENT BOOKLETS, PUZZLES AND GAMES

McGraw-Hill Book Co.,
Webster Division
Toronto

Exploring Mathematics on Your Own - ". . . is a fascinating series of enrichment booklets tailored for students who want to go beyond the textbook." (1961)

List of topics included in this series:

- Sets, sentences, and operations
- The Pythagorean Theorem
- Invitation to Mathematics
- Understanding Numeration Systems
- Fun With Mathematics
- Number Patterns
- Topology--the Rubber-Sheet Geometry
- The World of Measurement
- Adventures in Graphing
- Computing Devices

Spitzer, Herbert F., Activities for the Enrichment of Arithmetic (1964). "This handbook on arithmetic enrichment was prepared as one means of meeting the widespread demand on the part of teachers and parents for materials to use in stimulating and maintaining children's interest in mathematics during the elementary school years. . . . The extensive collection of materials in the handbook is organized by grades."

National Council of Teachers of Mathematics
1202 Sixteenth St., N.W.,
Washington, D.C. 20036

Booklets: Fujii, John N. , Puzzles and Graphs
Johnson, Donovan A., Paper Folding for the Mathematics Class
Peck, Lyman C., Secret Codes, Remainder Arithmetic and Matrices
Wenninger, Magnus J., Polyhedron Models for the Classroom.

Bibliographies:
Hardgrove and Miller, Mathematics Library - Elementary & Junior High School (1968)
Schaaf, William L., The High School Mathematics Library (1970)
________, A Bibliography of Recreational Mathematics (4th edition 1970)

J. Weston Walch
Portland, Maine

Anderson, R. Perry, Mathematical Bingo (1963).
"...presents a situation in which the pupil works a maximum number of exercises in a minimum amount of time."
- This book contains directions for playing mathematical bingo, 75 sheets of exercises, nearly 50 mathematical bingo cards, a master tally list, and cardboard squares. Since the exercises are grouped into 25 different categories (examples: integraters, linear equations and inequalities, complex numbers, geometry, other number bases, puzzlers), this game could be used for drill and for review at nearly every level.

Branden, Louis Grant, A Collection of Cross-Number Puzzles (1957)
"It is the purpose of the book to provide a collection of cross-number puzzles that can be used as a teaching aid with general mathematics classes."
This book contains puzzles that involve whole numbers, fractions, decimals, per cent, powers and square roots, measures, perimeters, areas, and volumes. The teacher section in this book contains chapters on such topics as: an introduction to cross-number puzzles as a teaching aid for secondary school pupils, reactions of teachers to the use of cross-number puzzles as a teaching device, a review of the literature pertaining to cross-number puzzles, and construction of the common cross-number puzzle.

____________, 4 the Math Wizard (1962)
"The book is different from most other publications on mathematics enrichment in that the items have been selected for use with certain secondary school students, collected in a single volume, and revised in keeping with the interests of students."
The teacher index "...is provided to serve as a guide for presenting some of the materials in this book to mathematics classes. Items are listed under different headings in a suggested sequence for class presentation."
The headings are: appreciation, logical reasoning, vocabulary, fundamentals, number theory, measure, general math, polygons and circles, perimeters, areas, and volumes, algebra, geometry, trigonometry, binaries, and probability.
This book also contains a 9-page bibliography of math enrichment publications.

____________, Yes, Math Can Be Fun! (1960)
"The materials in this book are the result of an attempt to collect the recreational mathematics items that have served to create interest and stimulate learning in the secondary school mathematics subjects."
This book covers the following topics: number oddities, puzzles, tricks and games, facts and stories, test yourself, illusions, some tough nuts to crack, and projects as well as an appendix containing a bibliography of recreational math publications and a teacher index for presenting materials to classes.

Johnson, Donnovan A., Games for Learning Mathematics (1960).
Level: junior high and high school, primarily.
Contents: card games, mathematical bingo, panel games, graphing and measurement games, mathematical tic-tac-toe, seasonal games (football, baseball, etc.), vocabulary practice, parlor games with new rules, relay contests, mathematics party games, elementary arithmetic games, and a list of commercial games.

Ransom, William R., One Hundred Mathematical Curiosities (1955).
"During a half-century of teaching mathematics, many problems that did not arise in class work have been brought to the author's attention over and over again....These problems form the basis for this mathematical museum which has been enlarged to include other matters that come up rather on the fringe of class work, whose treatment in more advanced books seems to be too much entangled with associated matter to be readily intelligible."
In the cross-referenced index, the topics are classified as follows: algebraic principles, algebraic problem statements, approximations, arithmetical, fallacious, geometric, familiar puzzles, number theory, integers required, polygons, tabulations, trick questions, and trigonometric.

Additional References

Carson & Armstrong, Thinking Through Mathematics, Thomas Nelson and Sons, Canada Limited, 1969.
"Mathematical ideas are handled in a spiral pattern. Fundamental concepts are introduced early, but are to be considered only within the limits of the understanding of the children. They are treated in a way that is consistent with what is to come later, care being taken to avoid anything that might have to be 'untaught' at higher levels. The fundamental concepts are reintroduced in succeeding stages to be dealt with in greater breadth and depth. As the children proceed through the various stages, it is expected that they will mature mathematically and maintain an interest in exploring further the field of mathematical thought."
The authors present the material in such a way that each concept or skill is carried through four phases: experience, understanding, accuracy, and facility. After listing some of the classroom equipment and material to be used, an overview of the program is given, grouping the material to be covered into the following headings: geometry; sets; numbers, numerals, and notation; number phrases and number sentences; mathematical operations, measurement; and graphs. Each section in the teacher's guide covers the following areas: concepts and vocabulary, developmental experience for pupils, suggestions for use of the pupil's book, and related activities.
Level: primary.

Court, Nathan A., Mathematics in Fun and Earnest, New York: The New American Library, 1958. Contents: Mathematics and Philosophy
Some Sociologic Aspects of Mathematics
The Lure of the Infinite
Mathesis the Beautiful (Mathematics and esthetics; art and mathematics).

Mathematics and the Mathematician
Mathematical Asides
Mathematics as Recreation

Gardner, Martin, Mathematical Puzzles and Diversions, New York: Simon and Schuster (Rockefeller Center, 630 Fifth Avenue, N.Y., N.Y. 10020), 1959. Contents: hexaflexagons, magic with a matrix, nine problems, ticktacktoe, probability paradoxes, the icosian game and the Tower of Hanoi, curious topological models, the game of hex, Sam Lloyd: America's greatest puzzlist, mathematical card tricks, memorizing numbers, polyominoes, fallacies, nim and tac tix, and left or right? (Also contains a list of references for further reading).

Longley-Cook, L.H., Work This One Out, London (England): Ernest Benn Ltd., (Bouverie House, Fleet Street), 1960. A book of mathematical problems.

Madachy, Joseph S., Mathematics on Vacation, New York: Charles Scribner's Sons, 1966. Contents: geometric dissections; chessboard placement problems; fun with paper; magic and antimagic squares; puzzles and problems; number recreations; alphametics; conglomerate; and a 3-page bibliography for further references.
"....ranges from brain teasers a novice can solve to sophisticated aspects of number theory."

Some Teacher-Created Drill Games and Graphing Games. (from Mathematics Seminar) ???

Bibliography from 2 booklets on elementary math games

Arithmetic Games and Activities, Wagner, Hosier and Gilloley, Teachers Publishing Corporation, Darien, Connecticut.

Games Make Arithmetic Fun, John F. Dean, Publisher, Newport Beach, California.

Mathematical Puzzles, Geoffrey Mott-Smith, Dover Publications

Magic House of Numbers, Irving Adler, John Day Company, Inc., Pub.

Mathematics -- Modern Concepts and Skills, Book 2, Dilley and Rucker, D.C. Heath.

Journal of Education, Boston University School of Education, 765 Commonwealth Avenue, Boston, Massachusetts 02115, Vol. 149, December, 1966.

Very Short Course in Mathematics for Parents, SMSG, A. C. Vroman, Inc., Pasadena California.

Life Science Library Mathematics, David Bergamini and Editors of Life, The Silver Burdett Co., 1963.

Review Tests Can Be Different, Louise Hasserd.

Exploring Arithmetic, Osborn, Rieflins, and Spitzer, Webster Division of McGraw-Hill.

ADDRESSES OF MANUFACTURERS OF COMMERCIAL CARD GAMES

James W. Lang,
P. O. Box 224,
Mound, Minn. 55364,
U. S. A.

Holt, Rinehart and Winston, Inc.,
383 Madison Avenue,
New York, N.Y. 10017,
U. S. A.

Krypto Corporation
2 Pine Street,
San Francisco, Calif. 94111,
U. S. A.

Milton Bradley Company,
74 Park Street,
Springfield, Mass. 01105,
U. S. A.

Ed-U-Cards Manufacturing Corporation,
Long Island City, New York,
U. S. A.

5. APPLICATION PROBLEMS IN THE HIGH SCHOOL CURRICULUM

Numbers under each topic refer to books listed on the last page:

I Relations and functions

1. Relations approached by starting with sets common to the students such as "...had....to drink yesterday." Relations used to introduce some graph theory.

3. Experiment: what is the largest overhang of a stack of books? This is developed into work with functions and graphing.

4. Experiments with a balance, loaded beam, falling sphere.

5. Experiments with absorption of liquids, horizontal metronome, oscillating spring, and inclined plans leading to work with non-linear functions.

9. Chapter 1: some illustrations of linear and non-linear functions--platform on rollers, pulley, algebraic balance.
   Chapter 4: solving equations by graphing; system of linear equations applied to chemistry
   Chapter 11: solving quadratics and applications to heredity
   Chapter 6: Circuits and springs.

12. Chapter 8

13. Chapter 4: rates of change
    Chapter 10

14. Chapter 3: the tangent function
    Chapter 4: logarithmic function
    Chapters 8 & 11: information from physical situations (rates of change)

II Analytic Geometry

9. Chapter 10

14. Chapter 12: vectors
    Chapter 9: three-dimensional geometry

III The Real Numbers

5. Loaded beam experiment

IV **Linear Algebra**

6. Matrix algebra used to develop transformations. This is applied to relativity theory.

8. Chapter 8: transformations through matrix algebra.

9. Chapter 4: System of linear equations applied to chemistry:

10. Chapter 2: geometric representations and translation in linear algebra.
    Chapter 3: mappings and matrices.

14. Chapters 1 & 2; matrices and inverses

V **Topology and Transformations**

1. Introduction to topology through investigation of line patterns. Geometry beginning with manipulation of objects (symmetry, ratios, angles).
   REFLECTION AND ROTATION LEADING TO WORK WITH VECTORS

6. Matrix algebra used to develop transformations. This is applied to relativity theory.

8. Chapter 6: interesting examples of non-Euclidean geometry.
   Chapter 8: Transformations through matrix algebra.
   Chapter 10: Projective geometry
   Chapter 12: Conformal transformations, transformations of quadratic equations and graphs.

9. Chapter 12: algebraic representation of some simple transformations.

10. Chapter 4: Transformations involved in oscillations--applications to difference and differential equations and to calculus in chapter 5.

11. Chapter 10: areas
    Chapter 13: symmetry

12. Chapter 1: topology. Chapter 3: similarity. Chapter 5: reflection and rotation. Chapter 7: translations. Chapter 10: dolids. Chapter 14: Pythagorean theorem.

13. Chapter 2: reflection, rotation, translation
    Chapter 3: matrix operations and transformations
    Chapter 5: geometry of the circle and volume of cylinder.
    Chapter 6: networks and matrices.
    Chapter 7: three-dimensional geometry.
    Chapter 12: the shearing transformation.
    Chapter 15: locus.

14. Chapters 1 & 5: transformations. Chapter 7: networks. Chapter 14: Geometry: conclusions from data.

VI Trigonometry

9. Chapter 10

12. Chapter 12

13. Chapter 9

VII Probability and Statistics

2. Muscle fatique experiments: statistics - mean, median, mode

9. Chapter 9: permutations.
   Chapter 13: finding correlation and spread from statistical information.

12. Chapter 2: Statistics (kinds of graphs)

13. Chapter 1: probability developed from experiments
    Chapter 2: Statistics

14. Chapter 6: applications of statistics
    Chapter 13: compound probabilities

VIII Calculus

6. The law of the lever applied to area under a parabola

10. Chapters 4 & 5: transformations involved in oscillations, and applications to calculus.
    Chapter 8: discussion of dericatives, minimum and maximum values, space with infinite dimensions.

IX Linear Programming

1. Comparison of objects (inequalities) leading to work with variables.

7. Pp. 22 and following: applications of linear programming to business.

13. Chapter 8

X Measurement

2. Measurement of lengths leading to a ratio and graphing
   Surface area and volume applied to biology.

3. Measurement of an object with different units leading to graphing and linear functions.

BOOK LIST

1. T. J. Fletcher, Some Lessons in Mathematics. (Cambridge University Press

2. School Mathematics Study Group. Mathematics and Living Things (Vrom ... California)

3. School Mathematics Study Group. Mathematics Through Science: Part I

4. School Mathematics Study Group, Mathematics Through Science, Part II

5. School Mathematics Study Group, Mathematics Through Science, Part III

6. School Mathematics Study Group. Studies in Mathematics Volume X. (Applied Mathematics in the High School)

7. The National Research Council, The Mathematical Sciences, (COSRIMS, M.I.T. Press)

8. W. W. Sawyer, Prelude to Mathematics, (Penguin)

9. W. W. Sawyer, The Search for Pattern (Penguin)

10. W. W. Sawyer, A PAth to Modern Mathematics. (Penguin)

11. School Mathematics Project. Text: Book 1 (Cambridge University Press)

12. School Mathematics Project, Text: Book 2 (Cambridge University Press)

13. School Mathematics Project. Text: Book 3 (Cambridge University Press)

14. School Mathematics Project. Text: Book 4 (Cambridge University Press) Available from the Macmillan Company, 70 Bond Street, Toronto.

Other Useful books and pamphlets

Ontario Institute for Studies in Education, Geometry: Kindergarten to 13 Toronto, O.I.S.E., 1967.

Mathematics Council of the Alberta Teachers' Association: Making Mathematics Practical. 1970 yearbook (ATA, Barnett House, Edmonton).

Mathematics Council of the Alberta Teachers' Association: Active Learning in Mathematics: a set of resource materials for teachers. 1971 yearbook (ATA, Barnett House, Edmonton)

School Mathematics Project, Book 5, Additional Book, Parts 1 and 2, Advanced Book 1.

Sawyer, W. W., Vision in Elementary Mathematics (Penguin). Particularly useful in junior high school, grades 7 - 9.

Sawyer, W. W., Mathematician's Delight (Penguin)
Sawyer, W.W., What is Calculus About? (Random House, paperback)
Vergara, Mathematics in Everyday Things (paperback)
Kramer, The Main Stream of Mathematics (paperback)
Friedrichs, From Pythagoras to Einstein (Random House, paperback)
Fisher, D., An Active Learning Unit on Real Numbers (ATA, Barnett House, Edmonton, 1971)

6. AN EVALUATION OF AN ACTIVE LEARNING UNIT ON REAL NUMBERS

"An active Learning Unit on Real Numbers", by Dale Fisher (available from Alberta Teachers' Association, Barnett House, Edmonton, Alberta).

This unit contains many activities aimed at "fostering an understanding of the concepts involved and promoting active learning on the part of the students". The teacher's task is guiding the student through the unit, judging as to whether or in what way any particular activity would be used. Games such as tic-tac-toe, battleship, point set game, property bee, chain puzzles, magic algebra, real number game, graphing pictures and how to locate pirates' gold or how to shoot the Red Baron form an important role in this unit. The purpose of the games and activities is to provide practice, enrichment, promote enthusiasm, and add interest to the different number systems. The teacher will have to give very careful individual guidance to ensure that each student does the type of activity suited for him.

While there are many useful ideas in the unit, serious questions can be raised. The impression that the unit gives is that students can take or leave any activity depending on whether they "like" it. We must be demanding of our students so that he develops his talents to the best of his ability, although at the same time we must search for ways of making the material meaningful for the student. Schools should be pleasant, and the students must be in such an atmosphere that they have the courage to tackle problems and activities without being "scared". At the same time, students should be faced with difficult problems with which they have to struggle, and not be told that they can give up whenever they don't like something.

We can also raise questions about the place that Fisher gives to the real numbers in the curriculum. He stresses the computation of square roots out of all proportion. He introduces the real numbers through the artificial notion of infinite decimals - students will not grasp the necessity for introducing real numbers from such an approach. The best approach (by means of the Pythagorean theorem; see the section on the history of mathematics) is mentioned only as an optional topic. We also question the need for a whole unit on the real numbers at this point: the important concepts can be developed more naturally in other contexts. Students are faced with irrational numbers and a real number line without first having been convinced that they are needed; in fact, most students probably see no necessity of their introduction at the end of the unit. The abstract approach is harmful for students with scientific or practical interests. With these students - in fact, with all students - we continually have to demonstrate that the mathematics we're doing is interesting and relevant to their concerns.

On the other hand, there are many ideas here that could be incorporated in some areas of the curriculum, and the unit is useful as a source book.